Same Same But Different — An Alphabetically Innocent Compositional Predicate Logic

Udo Klein & Wolfgang Sternefeld

“Same Same But Different” is the title of a movie by Detlev Buck, based on Benjamin Prüfer’s novel *Wohin du auch gehst. Die Geschichte einer fast unmöglichen Liebe*. We would like to thank Marcus Kracht . . . for comments and valuable proposals.

Das Poblem

- Prädikatenlogik ist in ihrer Standardformulierung, in der alle Formeln Wahrheitswerte bezeichnen, nicht kompositional.
- Eine PL, in der alle Formeln (partielle) Belegungen bezeichnen (also Folgen von Zuordnungen), ist zwar kompositional, aber nicht alphabetisch "'unschuldig" oder "'invariant": D.h. offene Formeln wie $F(x)$ und $F(y)$ haben verschiedene Extensionen. Dies aber ist konzeptuell unerwünscht.

Das Problem wurde von (Fine, 2007, 7) als "Russell's antinomy of the variable" bezeichnet:

"Russell's antinomy of the variable" (cf. (Fine, 2007, 7)):

Suppose that we have two variables, say "x" and "y"; and suppose that they range over the same domain of individuals, say the domain of all real numbers. Then it appears as if we wish to say contradictory things about their semantic role. For when we consider their semantic role in two distinct expressions – such as "$x > 0$" and "$y > 0$," we wish to say that it is the same. Indeed, this would appear to be as clear a case as one could hope to have of a merely "conventional" or "notational" difference; the difference is merely in the choice of the symbol and not at all in linguistic function. On the other hand, when we consider the semantic role of the variables in the same expression – such as in "$x > y$" – then it seems equally clear that it is different. Indeed, it would appear to be essential to the semantic role of the expression as a whole that it contains two distinct variables, not two occurences of the same variable, and presumably this is because the roles of the distinct variables are not the same.

Zusammenfassung

- Ziel:
  eine Logiksprache (mit Variablen), die sowohl kompositional interpretierbar als auch alphabetisch invariant ist.
- Ausgangshypothese: Die Denotation offener Formeln sind nicht Wahrheitswerte, sondern Mengen von (endlichen) Folgen, die aber nicht von Belegungen abhängen, sondern von den internen (syntaktischen) Beziehungen der Variablen untereinander.
- Annahme: Die Länge dieser Folgen stimmt mit der Anzahl der Vorkommen von Variablen in der Formel überein.

Zusammenfassung

- Idee 1:
  Folgen von Variablen (sog. K-Sequenzen) haben eine eigene Denotation. D.h. die Extension einer Formel $P(x, y, \ldots z)$ berechnet sich aus der Extension von P und der Extension der Folge $\langle x, y, \ldots z\rangle$. Die Extension einer Folge kodiert idealerweise lediglich, ob zwei Zeichen (Variablen) in der Folge gleich oder verschieden sind.
- Konsequenzen:
  $[\![x, y]\!] = [\![y, z]\!] = [\![x, z]\!] = \ldots$
  UND:
  $[\![x, x]\!] = [\![y, y]\!] = [\![z, z]\!] = \ldots$
  ABER:
  $[\![x, x]\!] \neq [\![x, y]\!]$

Zusammenfassung

- Idee 2:
  Jede Verknüpfung von 2 K-Sequenzen zweier Formeln zu einer komplexeren Formel benötigt ein sog. Koordinationsschema, später auch Desambiguierung genannt, d.h. eine Aussage darüber, welche Variablen der beiden Formeln in der neuen Formel zu identifizieren sind.
- Beispiel 1:
  Pyx disjunktiv verknüpft mit Qyx mit dem Koordinationsschema $\langle a, b, c, a \rangle$ ergibt: $Pyx \vee Qzy$.
  Die Extension der komplexen Formel ergibt sich dann aus $[\![\vee]\!]$, $[\![P]\!]$, $[\![Q]\!]$ und $[\![abca]\!]$.
  Notationskonvention: Wir schreiben das Koordinationsschema direkt in die Formel und benutzen eine Art "‘polnische Notation”: $\vee PQ.abca$

Zusammenfassung

- Beispiel 2:
  $Pxyz$, soll verküpft werden mit $\exists x$
  Das Koordinationsschema *abca* würde für $\exists xP.xyz$ nun besagen, dass z abgebunden wird.
- Möglichkeiten, dies zu notieren:
  - Aritätsvermehrend: $\exists P.abca$
    Die Bedeutung ergibt sich kompositional aus $[\![\exists]\!]$, $[\![P]\!]$ und $[\![abca]\!]$.
  - Aritätsvermindernd: $\exists abcaP.xy$
    Die Bedeutung ergibt sich kompositional aus $[\![\exists]\!]$, $[\![P]\!]$ und $[\![abca]\!]$.
  - Aritätskonstant: $\exists bcaP.bc\underline{a}$
    Hierbei muss mit der Unterstreichung die Idee der gebundenen Variablen eingeführt werden. Die Bedeutung ergibt sich aus $[\![\exists]\!]$, $[\![P]\!]$ und der Differenz zwischen $[\![bca]\!]$ und $[\![bc\underline{a}]\!]$

Überblick

- Einf. sog. ϵ/ν-Strukturen als Extension von K-Sequenzen
- Kompositionale Berechnung von ϵ/ν-Strukturen
- Koordinationsschemata (ambiguous merge)
- Normierte Darstellungen von K-Sequenzen
- Aritätserhaltende Prädikatenlogik
- Aritätsreduzierene PL
- Aritätsvermehrende PL
- Großer Anwendungsteil (nicht heute!)

Same or different: ϵ/ν-Structures

(1) **Definition ($\delta, \epsilon/\nu$-pairs):**
Let $z = \langle b_1, \ldots b_n \rangle$. Define $\delta(\langle i, j \rangle, z)$ as that function that assigns to a pair of positions $\langle i, j \rangle$, $1 \leq i, j \leq n$, and a sequence z the pair $\langle\langle i, j \rangle, \epsilon\rangle$ if $b_i = b_j$, and $\langle\langle i, j \rangle, \nu\rangle$ otherwise. Pairs of the form $\langle\langle i, j \rangle, x\rangle$ where $x \in \{\epsilon, \nu\}$ are called ϵ/ν-pairs.

(2) **Definition (t-equivalence):**
Two n-ary sequences z_1 and z_2 are t-equivalent iff for all positions i, j with $1 \leq i, j \leq n$: $\delta(\langle i, j \rangle, z_1) = \delta(\langle i, j \rangle, z_2)$. If z_1 and z_2 are t-equivalent, we write $z_1 =_t z_2$.

Same or different: ϵ/ν-Structures

To illustrate, the sequence of variables $x_1 x_4 x_1 x_2 x_1$ should be assigned the denotation κ:

$$
\begin{array}{llllll}
\kappa = \{ & \langle\langle 1,1\rangle,\epsilon\rangle, & \langle\langle 2,1\rangle,\nu\rangle, & \langle\langle 3,1\rangle,\epsilon\rangle, & \langle\langle 4,1\rangle,\nu\rangle, & \langle\langle 5,1\rangle,\epsilon\rangle, \\
 & & \langle\langle 2,2\rangle,\epsilon\rangle, & \langle\langle 3,2\rangle,\nu\rangle, & \langle\langle 4,2\rangle,\nu\rangle, & \langle\langle 5,2\rangle,\nu\rangle, \\
 & & & \langle\langle 3,3\rangle,\epsilon\rangle, & \langle\langle 4,3\rangle,\nu\rangle, & \langle\langle 5,3\rangle,\epsilon\rangle, \\
 & & & & \langle\langle 4,4\rangle,\epsilon\rangle, & \langle\langle 5,4\rangle,\nu\rangle, \\
 & & & & & \langle\langle 5,5\rangle,\epsilon\rangle \quad \}
\end{array}
$$

(3) **Definition (ϵ/ν-structure):**
κ is an ϵ/ν-structure of rank n iff κ is a function from all pairs of integers i, j with $1 \leq j \leq i \leq n$ into the set $\{\epsilon, \nu\}$ such that for all i, j, k:

a. Reflexivity: $\kappa(i,i) = \epsilon$,

b. Transitivity: if $\kappa(i,j) = \epsilon$ and $\kappa(j,k) = \epsilon$, then $\kappa(i,k) = \epsilon$.

Same or different: ϵ/ν-Structures

(4) **Definition (*i*-th projection):**
Define $\pi_i(s)$ as the i-th element of a sequence s, also called the i-th projection of s.

(5) If σ is an n-tuple of symbols, then: (inadäquate Definition!)

$$[\![\sigma]\!] := \{\langle\langle i,j\rangle,\epsilon\rangle : 1 \leq j \leq i \leq n \wedge \pi_i(\sigma) = \pi_j(\sigma)\} \cup \{\langle\langle i,j\rangle,\nu\rangle : 1 \leq j \leq i \leq n \wedge \pi_i(\sigma) \neq \pi_j(\sigma)\}$$

(6) **Definition (t respects κ):**
An n-tuple $t = \langle a_1, \ldots, a_n\rangle$ respects an ϵ/ν-structure κ of rank n iff

$$\forall i \forall j[1 \leq i,j \leq n \rightarrow (\kappa(i,j) = \epsilon \rightarrow x_i = x_j)]$$

(7) If $I(R)$ is the denotation of an n-place relation R and if $[\![\sigma]\!]$ is an ϵ/ν-structure of rank n, then $[\![R\sigma]\!] = \{s \in I(R) : s \text{ respects } [\![\sigma]\!]\}$.

Kompositionale Definition von K-Sequenzen

(8) **Definition (K-sequence, still inadequate):**
Let K be a denumerably infinite set of symbols. Then:

a. Empty string:
The empty string Λ is a K-sequence of arity 0 (the empty K-sequence).

b. Base case:
If σ is an n-ary K-sequence and X is a symbol in K not occuring in σ, then $f_b(\sigma, X) = \sigma X$ is an $n+1$-ary K-sequence.

c. Reflexivisation:
If σ is an n-ary K-sequence with $\pi_i(\sigma) = X$, then $f_{r_i}(\sigma, X) = \sigma X$

Die Semantik wäre kompositional, aber wir brauchen unendlich viele Operationen. Es folgt die Reduzierung auf endlich viele...

Kompositionale Definition von K-Sequenzen

(9) **Definition (K-sequence):**
Let K be a denumerably infinite set of symbols. Then:

a. Empty string:
The empty string Λ is a K-sequence of arity 0 (the empty K-sequence).

b. Base case:
If σ is an n-ary K-sequence and X is a symbol in K not occuring in σ, then $f_b(\sigma, X) = \sigma X$ is an $n+1$-ary K-sequence.

c. Conservative extension:
If $X_1 \ldots X_n$ is an n-ary K-sequence, then $f_{ce}(X_1 \ldots X_n) = X_1 \ldots X_n X_n$ is an $n+1$-ary K-sequence.

d. Long conversion:
...

Kompositionale Definition von K-Sequenzen

(9) a. (i) Long conversion:

x_9	x_1	x_7	x_7
$\langle\langle 1,1\rangle,\epsilon\rangle$	$\langle\langle 2,1\rangle,\nu\rangle$	$\langle\langle 3,1\rangle,\nu\rangle$	$\langle\langle 4,1\rangle,\nu\rangle$
	$\langle\langle 2,2\rangle,\epsilon\rangle$	$\langle\langle 3,2\rangle,\nu\rangle$	$\langle\langle 4,2\rangle,\nu\rangle$
		$\langle\langle 3,3\rangle,\epsilon\rangle$	$\langle\langle 4,3\rangle,\epsilon\rangle$
			$\langle\langle 4,4\rangle,\epsilon\rangle$

$\xrightarrow{LC}$

(ii)

x_1	x_7	x_7	x_9
$\langle\langle 1,1\rangle,\epsilon\rangle$	$\langle\langle 2,1\rangle,\nu\rangle$	$\langle\langle 3,1\rangle,\nu\rangle$	$\langle\langle 4,1\rangle,\nu\rangle$
	$\langle\langle 2,2\rangle,\epsilon\rangle$	$\langle\langle 3,2\rangle,\epsilon\rangle$	$\langle\langle 4,2\rangle,\nu\rangle$
		$\langle\langle 3,3\rangle,\epsilon\rangle$	$\langle\langle 4,3\rangle,\nu\rangle$
			$\langle\langle 4,4\rangle,\epsilon\rangle$

(10) $$g(i) = \begin{cases} n, & \text{if } i = 1 \\ i-1, & \text{otherwise} \end{cases}$$

Kompositionale Definition von K-Sequenzen

(11)

x_1	x_7	x_7	x_9
$\langle\langle 1,1\rangle,\epsilon\rangle$	$\langle\langle 2,1\rangle,\nu\rangle$	$\langle\langle 3,1\rangle,\nu\rangle$	
	$\langle\langle 2,2\rangle,\epsilon\rangle$	$\langle\langle 3,2\rangle,\epsilon\rangle$	
		$\langle\langle 3,3\rangle,\epsilon\rangle$	
$\langle\langle 1,4\rangle,\nu\rangle$	$\langle\langle 2,4\rangle,\nu\rangle$	$\langle\langle 3,4\rangle,\nu\rangle$	$\langle\langle 4,4\rangle,\epsilon\rangle$

(12) $NORM(A) := \{\langle\langle i,j\rangle,x\rangle : i \geq j \wedge (\langle\langle i,j\rangle,x\rangle \in A \vee \langle\langle j,i\rangle,x\rangle \in A\}$

(13) Semantics for long conversion:
$f_{lc}([\![\sigma]\!]) := NORM(\{\langle\langle g(i),g(j,)\rangle,x\rangle : \langle\langle i,j\rangle,x\rangle \in \sigma\})$

(14) Swap of positions for short conversion:

$$f(i) = \begin{cases} i+1, & \text{if } i = n-1 \\ i-1, & \text{if } i = n \\ i, & \text{otherwise} \end{cases}$$

Kompositionale Bedeutung von K-Sequenzen

(15) **Definition (meaning of a K-sequence):**

a. Empty sequence: wie oben
b. Base case: wie oben
c. Conservative extension: wie oben
d. Long conversion:
Let $\sigma' = f_l(\sigma)$ be an n-ary K-sequence. Then:

$$f_{lc}([\![\sigma']\!]) := NORM(\{\langle\langle g(i), g(j,)\rangle, x\rangle : \langle\langle i, j\rangle, x\rangle \in \sigma\})$$

e. Short conversion:
Let $\sigma' = f_s(\sigma)$ be an n-ary K-sequence. Then:

$$f_{sc}([\![\sigma']\!]) := NORM(\{\langle\langle f(i), f(j,)\rangle, x\rangle : \langle\langle i, j\rangle, x\rangle \in \sigma\})$$

Ambiguous Merge

(16) **Definition (K-sequence, Merge):**

a. Let K be a denumerably infinte set of symbols (eg. $\{\bigcirc, \square, \diamond, \triangledown, \triangle \ldots\}$). The concatenation of n elements of K is called an n-ary K-sequence.[1]

b. Let σ_1 and σ_2 be K-sequences. Then the ambiguous merge $\sigma_1@\sigma_2$ is the set of all K-sequences $\sigma_1'\sigma_2'$ such that $\sigma_1' =_t \sigma_1$ and $\sigma_2' =_t \sigma_2$.

Zur Erinnerung: $=_t$ wurde in (2) definiert.

(17) **Definition (normalized sequence):**

Let W be an infinite sequence in K^* such that for all $i \neq j, \pi_i(W) \neq \pi_j(W)$. A K-sequence $\sigma \in K^n$ is normalized (with respect to W) if the following holds:

if $\pi_i(\sigma) = \pi_j$(W), $n \geq j > 1$, then π_{j-1}(W) = $\pi_k(\sigma)$ for some $k < i$.

[1] Unlike the standard notation $x_0, x_1, x_2, \ldots$ for variables, our notation has the slight advantage of imposing no internal structure to the symbols representing variables. A normalized K-sequences is what is called a Kenogramm in Mahler (1993).

Ambiguous Merge

(18) W = ⟨○, □, ◇, ▽, △ . . .⟩

(19) length 0: ⟨⟩ (= Λ)
length 1: ○
length 2: ○○, ○□
length 3: ○○○, ○○□, ○□○, ○□□, ○□◇
length 4: ○○○○, ○○○□, ○○□○, ○○□□, ○○□◇, ○□○○, ○□○□, ○□○◇, ○□□○, ○□□□, ○□□◇, ○□◇○, ○□◇□, ○□◇◇, ○□◇▽.
length 5: etc.

(20)

○□@○□	normalized *K*-seq.	positions identified	example
1	○□◇▽	–	△◇▽□
2	○□○◇	1-3	△□△▽
3	○□◇○	1-4	□▽△□
4	○□□◇	2-3	◇□□△
5	○□◇□	2-4	▽◇○◇
6	○□○□	1-3, 2-4	△○△○
7	○□□○	1-4, 2-3	◇○○◇

Ambiguous Merge

(21) **Definition (coordination):**
Assume that $\sigma_1 \in K^m, \sigma_2 \in K^n$ and that $\sigma_1' \sigma_2'$ is a disambiguation of $\sigma_1 @ \sigma_2$. A position i in σ_1' is coordinated with a position j in σ_2', iff $\pi_i(\sigma_1') = \pi_j(\sigma_2')$.

An alphabetically innocent, compositional, but still inadequate variant

(22) **Syntax:**

a. Definition of n-ary K-sequences as in (9).

b. Let R be a predicate constant of arity n, $R \notin K$, and σ a K-sequence. Then $f(R, \sigma) = R\sigma$ is a well-formed formula (WFF) of arity n, with R being its prefix and σ its R-profile.

c. If α and β are WFFs of arity n and m, with P and Q the prefixes and σ_1 and σ_2 the R-profiles of α and β respectively, and $\sigma_1'\sigma_2' \in \sigma_1 @ \sigma_2$ a disambiguation of $\sigma_1 @ \sigma_2$, then $f_\wedge(P, \sigma_1, Q, \sigma_2, \sigma_1'\sigma_2') = \wedge PQ\sigma_1'\sigma_2'$ is a WFF of arity $n + m$, where $\wedge PQ$ is its prefix, and $\sigma_1'\sigma_2'$ its R-profile.

d. If α is a WFF of arity n, with P its prefix, then $f_\neg(\alpha) = \neg\alpha$ is a WFF of arity n with $\neg P$ as its prefix.

e. If $\phi = P\sigma$ is a WFF of arity n with P its prefix and σ its R-profile, k is a symbol in K, and $k\sigma' \in K^{n+1}$ a disambiguation of $k@\sigma$, then the result of applying the existential rule $f_\exists$ to $P, \sigma,$ and $k\sigma'$ $f_\exists(P, \sigma, k\sigma') = \exists k\sigma' P\sigma'$ is an WFF of arity n, with $\exists k\sigma' P$ its prefix and σ' its R-profile.

Variants of predicate logic

(23) **Semantics:**
Let I be an interpretation of the relational constants of L.

a. $[\![\sigma]\!]$ as in (15)

b. $[\![R]\!] = I(R)$ for all constants R.

c. $[\![f(R,\sigma)]\!] = \mathcal{O}([\![R]\!],[\![\sigma]\!])$
$= \{s \in [\![R]\!] : s \text{ respects } [\![\sigma]\!]\}$

d.
$[\![f_\wedge(P,\sigma_1,Q,\sigma_2,\sigma_1'\sigma_2')]\!] = \mathcal{O}_\wedge([\![P]\!],[\![\sigma_1]\!],[\![Q]\!],[\![\sigma_2]\!],[\![\sigma_1'\sigma_2']\!])$
$= \{s \in [\![P\sigma_1]\!] \times [\![Q\sigma_2]\!] : s \text{ respects } [\![\sigma_1'\sigma_2']\!]\}$

e. $[\![f_\neg(\alpha)]\!] = \mathcal{O}_\neg([\![\alpha]\!])$.
$= D^n \setminus [\![\alpha]\!]$ for α with arity n

f. $[\![f_\exists(P,\sigma,k\sigma')]\!] = \mathcal{O}_\exists([\![P]\!],[\![\sigma]\!],[\![k\sigma']\!])$
$= \{s \in D^n : \exists s'.(s' \in [\![P\sigma]\!] \wedge$ for all i
if $\pi_i(s) \neq \pi_i(s')$ then $\langle\langle 1, i+1\rangle, \epsilon\rangle \in [\![k\sigma']\!]))\}$

Variants of predicate logic

(24) A WFF ϕ of arity n is

a. **true** iff $[\![\phi]\!] = D^n$

b. **false** iff $[\![\phi]\!] = \emptyset$,

c. **satisfiable** iff $[\![\phi]\!] \neq \emptyset$.

(25) A formula α **entails** β iff for all interpretations I: if α is true in I, then β is true in I.

DAS PROBLEM:

If $[\![\exists\bigcirc\bigcirc P\bigcirc]\!] = [\![\exists\bigcirc\bigcirc Q\bigcirc]\!] = D$ (where D is the domain of the model), the conjunction $[\![\wedge\exists\bigcirc\bigcirc P\exists\bigcirc\bigcirc Q\bigcirc\square]\!] = D^2$ yields a true sentence, as one would expect. However, this is not the case if we conjoin the two formulas using the a disambiguation that coordinates the two variables. If $D = \{a, b\}, I(P) = \{a\}, I(Q) = \{b\}$, then $[\![\wedge\exists\bigcirc\bigcirc P\exists\bigcirc\bigcirc Q\bigcirc\bigcirc]\!] = \{s : s \in [\![\exists\bigcirc\bigcirc P\bigcirc]\!] \times [\![\exists\bigcirc\bigcirc Q\bigcirc]\!] \wedge \pi_1(s) = \pi_2(s)\} = \{\langle a, a\rangle, \langle b, b\rangle\} \neq D^2$.

Arity preserving predicate logic (APPL)

(26) **Definition (K-symbol):**
Let K_F and K_B be denumerably infinite disjoint set of symbols (eg. $\{\bigcirc, \square, \diamond, \triangledown, \triangle, \ldots\}$ and $\{\bullet, \blacksquare, \blacklozenge, \blacktriangledown, \blacktriangle, \ldots\}$). A K-symbol is an element of $K_F \cup K_B$.

(27) a. First base clause:
If σ is an n-ary K-sequence and $X \in K_F$ is a K-symbol not occuring in σ, then $f_{b_1}(\sigma, X) = \sigma X$ is an $n+1$-ary K-sequence.

b. Second base clause:
If σ is an n-ary K-sequence and $X \in K_B$ is a K-symbol not occuring in σ, then $f_{b_2}(\sigma, X) = \sigma X$ is an $n+1$-ary K-sequence.

Arity preserving predicate logic (APPL)

- the newly built formulas are interpreted with respect to appropriate (modified) ϵ/ν-structures that reflect the difference between K_F and K_B;
- binding by a quantifier introduces a new (black) symbol that cannot be coordinated by any further operation.

(28) **Definition (ϵ/ν-structure, revised):**
κ is an ϵ/ν-structure of rank n iff κ is a function from all pairs of integers i, j with $1 \leq j \leq i \leq n$ into the set $\{\epsilon, \nu\}$ such that for all i, j, k:

a. Coordination of positions implies identity of color:
if $\kappa(i, j) = \epsilon$, then $\kappa(i, i) = \kappa(j, j)$,

b. Transitivity:
if $\kappa(i, j) = \epsilon$ and $\kappa(j, k) = \epsilon$, then $\kappa(i, k) = \epsilon$.

Arity preserving predicate logic (APPL)

(29) a. First base case:
Let $\sigma' = f_{b_1}(\sigma, X)$ be an n-ary K-sequence.
Then $[\![\sigma']\!] = [\![\sigma]\!] \cup \{\langle\langle n, i\rangle, \nu\rangle : 1 \leq i < n\} \cup \{\langle\langle n, n\rangle, \epsilon\rangle\}$

b. Second base case:
Let $\sigma' = f_{b_2}(\sigma, X)$ be an n-ary K-sequence.
Then $[\![\sigma']\!] = [\![\sigma]\!] \cup \{\langle\langle n, i\rangle, \nu\rangle : 1 \leq i < n\} \cup \{\langle\langle n, n\rangle, \nu\rangle\}$

(30) **Definition (binding):**
Assume $\sigma_1'\sigma_2'$ is a disambiguation of $\sigma_1 @ \sigma_2$, and a position i in σ_1' is coordinated with a position j in σ_2'. Then i in σ_1' binds j in σ_2' iff

a. $\pi_i(\sigma_1) \in K_B$,
b. $\pi_j(\sigma_2) \in K_F$, and
c. $\pi_j(\sigma_2') \in K_B$.

Arity preserving predicate logic (APPL)

(31) **Definition (ambiguous conjunction merge):**
Let σ_1 and σ_2 be two m- and n-ary K-sequences ($m > 0, n > 0$). Then the **ambiguous conjunction merge** $\sigma_1 @_c \sigma_2$ is the set of all K-sequences $\sigma_1'\sigma_2' \in \sigma_1 @ \sigma_2$ satisfying the following conditions:

a. $\forall i.(\pi_i(\sigma_1) \in K_F \leftrightarrow \pi_i(\sigma_1') \in K_F)$ (no change of "color" in σ_1),
b. $\forall i.(\pi_i(\sigma_2) \in K_F \leftrightarrow \pi_i(\sigma_2') \in K_F)$ (no change of "color" in σ_2),
c. $\neg\exists i \exists j.(\pi_i(\sigma_1') \in K_B \wedge \pi_j(\sigma_2') \in K_B \wedge \pi_i(\sigma_1') = \pi_j(\sigma_2'))$ (no coordination of bound variables).

(32) **Definition (ambiguous binding merge):**
Let $k \in K_B$, and let σ_2 be an n-ary K-sequence ($n > 0$). Then the ambiguous binding merge $k @_b \sigma_2$ is the set of all K-sequences $\sigma_1'\sigma_2' \in k @ \sigma_2$ satisfying the following condition:

a. $\exists j.$(1 in σ_1' binds j in σ_2') (k targets and binds an element in σ_2)
b. $\forall j.$((1 in σ_1' does not bind j in σ_2') $\rightarrow (\pi_j(\sigma_2) \in K_F \leftrightarrow \pi_j(\sigma_2') \in K_F)$) (the color of all other variables remains unchanged)

Arity preserving predicate logic (APPL)

(33) **Syntax of APPL:**

a. Definition of n-ary K-sequences as in (22-a) (=(9)).

b. Definition of f as in (22-b).

c. Definition of $f_\wedge$ as in (22-c) except that @ in (22-c) is now replaced by $@_c$.

d. Definition of $f_\neg$ as in (22-d).

e. If

(i) $P\sigma$ is a WFF of arity n with P its prefix and σ its R-profile,

(ii) $k \in K_B$, and c a constant

(iii) $k\sigma'$ is a disambiguation of $k@_b\sigma$,

then

$$f_\exists(P, \sigma, k\sigma') = \exists\sigma P\sigma'$$

$$f_{con}(c, P, \sigma, k\sigma') = c\sigma P\sigma'$$

are WFFs of arity n, with $\exists\sigma P$ and $c\sigma P$ as prefixes and σ' as R-profile.

Arity preserving predicate logic (APPL)

(34) **Semantics of APPL**:

a. Denotation of K-sequences as in (15) modified by replacing the base clause with (27).

b. $[\![R]\!] = I(R)$ for all constants R.

c. $[\![f(R,\sigma)]\!] = \mathcal{O}([\![R]\!],[\![\sigma]\!]) = \{s \in [\![R]\!] : s \text{ respects } [\![\sigma]\!]\}$

d. $[\![f_\wedge(P,\sigma_1,Q,\sigma_2,\sigma_1'\sigma_2')]\!] = \mathcal{O}_\wedge([\![P]\!],[\![\sigma_1]\!],[\![Q]\!],[\![\sigma_2]\!],[\![\wedge]\!])$
$= \{s \in [\![P\sigma_1]\!] \times [\![Q\sigma_2]\!] : s \text{ respects } [\![\sigma_1'\sigma_2']\!]\}$

e. $[\![f_\neg(\alpha)]\!] = \mathcal{O}_\neg([\![\alpha]\!]) = D^n \setminus [\![\alpha]\!]$ for α with arity n

f. $$\begin{aligned}[\![f_\exists(P,\sigma,k\sigma')]\!] &= \mathcal{O}_\exists([\![P]\!],[\![\sigma]\!],[\![k\sigma']\!])\\ &= \{s \in D^n : \exists s' \in [\![P\sigma]\!] \text{ and for all } i:\\ &\quad (\pi_i(s) \neq \pi_i(s') \to (\langle\langle i,i\rangle,\epsilon\rangle \in [\![\sigma]\!] \wedge\\ &\quad \langle\langle i+1,i+1\rangle,\nu\rangle \in [\![k\sigma']\!]))\}\end{aligned}$$

g. $$\begin{aligned}[\![f_{con}(c,P,\sigma,k\sigma')]\!] &= \mathcal{O}_{con}([\![c]\!],[\![P]\!],[\![\sigma]\!],[\![k\sigma']\!])\\ &= \{s \in D^n : \exists s' \in [\![P\sigma]\!] \text{ and for all } i:\\ &\quad (\pi_i(s) \neq \pi_i(s') \to \langle\langle i,i\rangle,\epsilon\rangle \in [\![\sigma]\!] \wedge\\ &\quad \langle\langle i+1,i+1\rangle,\nu\rangle \in [\![k\sigma']\!] \wedge \pi_i(s') = I(c)))\}\end{aligned}$$

Arity reducing predicate logic (ARPL)

(35) **Definition (1-reduction)**:
Let $\sigma = \langle x_1, \ldots, x_n \rangle$ be a K-sequence with $n \geq 1$, Λ the empty string, and + concatenation. Then the 1-reduction $r_1(\sigma)$ is defined as follows:

$$r_1(\langle x_1, \ldots, x_n \rangle) = \begin{cases} \Lambda, & \text{if } n = 1 \\ r_1(\langle x_1, \ldots, x_{n-1} \rangle), & \text{if } n > 1 \wedge x_1 = x_n \\ r_1(\langle x_1, \ldots, x_{n-1} \rangle) + x_n, & \text{if } n > 1 \wedge x_1 \neq x_n \end{cases}$$

Beispiel:

(36)
$$\begin{aligned} r_1(\bigcirc\Box\bigcirc\Diamond) &= r_1(\bigcirc\Box\bigcirc)\Diamond && \text{(third clause)} \\ &= r_1(\bigcirc\Box)\Diamond && \text{(second clause)} \\ &= r_1(\bigcirc)\Box\Diamond && \text{(third clause)} \\ &= \Box\Diamond && \text{(first clause)} \end{aligned}$$

Arity reducing predicate logic (ARPL)

(37) **Syntax of existential quantification:**

a. Let $P\sigma$ be a WFF of arity n with P its prefix and σ its R-profile,
b. $k \in K$,
c. $k\sigma'$ a disambiguation of $k@\sigma$, and
d. $\sigma'' = r_1(k\sigma')$,

then

$$f_{\exists}(P, \sigma, k\sigma') = \exists k\sigma' P\sigma''$$

is a WFF of arity m, with $\exists k\sigma' P$ its prefix, σ'' its R-profile, and m the arity of σ''.

Arity reducing predicate logic (ARPL)

(38) **Definition (sentence):**
A WFF ϕ is a sentence iff its arity is 0, i.e. its R-profile is the empty string Λ.

(39) **Definition (true sentence):** [2]
A sentence ϕ is true iff $[\![\phi]\!] = \{\emptyset\} = D^0$.

[2] The following should hold for all *n*-ary WFF in an arity reduction semantics:

$$[\![\neg\phi]\!] = \begin{cases} \{\emptyset\}\backslash[\![\phi]\!], & \text{if } \sigma = \Lambda \\ D^n\backslash[\![\phi]\!], & \text{if } \sigma \neq \Lambda \end{cases}$$

This follows from our previous definition of negation if we may assume, as we do in (39), that $D^0 = \{\emptyset\}$.

Arity reducing predicate logic (ARPL)

(40) **Definition (κ-reduction)**:
Let κ be an ϵ/ν-structure of rank n and s a sequence of length i, $1 \leq i \leq n$. Then

$$r_\kappa(s) = \begin{cases} \emptyset & \text{if } i = 1 \\ r_\kappa(\langle \pi_1(s), \ldots, \pi_{i-1}(s) \rangle), & \text{if } i > 1 \wedge \kappa(1,i) = \epsilon \\ r_\kappa(\langle \pi_1(s), \ldots, \pi_{i-1}(s) \rangle) + \pi_i s), & \text{if } i > 1 \wedge \kappa(1,i) = \nu \end{cases}$$

(41) $[\![f_\exists(P, \sigma, k\sigma')]\!] = \{t : \exists s.(s \in D \times [\![P\sigma]\!] \wedge t = r_{[\![k\sigma']\!]}(s))\}$

Arity reducing predicate logic (ARPL)

(42) **Introduction of constants in ARPL**

a. Syntax:
Let $P\sigma$ be a WFF, let c be an individual constant, and let $k\sigma' \in k@\sigma$. Then

$$f_{con}(c, P, \sigma, k\sigma') = ck\sigma' P\sigma''$$

is a WFF of rank n, where $\sigma'' = r_k(\sigma')$ and n is the length of σ''.

b. Semantics:
$[\![f_{con}(c, P, \sigma, k\sigma')]\!] = \{t : \exists s.s \in D \times [\![P\sigma]\!] \wedge \forall i([\![k\sigma']\!](1, i) = \epsilon \rightarrow \pi_i(s) = I(c)) \wedge t = r_{[\![k\sigma']\!]}(s)\}$

Arity-increasing predicate logic

(43) **Syntax of quantification:**

a. Let $P\sigma$ be a WFF of arity n with P its prefix and σ its R-profile,
b. $k \in K_B$, and
c. $k\sigma'$ a disambiguation of $k@_b\sigma$,

then $\exists Pk\sigma'$ is a WFF of arity $n+1$, with $\exists P$ its prefix and $k\sigma'$ its R-profile.

(44) **Semantics of quantification:**

$$[\![f_\exists(P,\sigma,k\sigma')]\!] = \{s \in D^{n+1} : \exists s' \in D \times [\![P\sigma']\!],\ \text{and for all } i : ([\![k\sigma']\!](1,i) = \nu \rightarrow \pi_i(s) = \pi_i(s'))\}$$

Fine, Kit (2007). *Semantic Relationalism*. Blackwell, Oxford.

Mahler, Thomas (1993). *Morphogrammatik. Eine Einführung in die Theorie der logischen Form*. www.thinkartlab.com/pkl/tm/MG-Buch.pdf.

Quine, Willard Van Orman (1960). ‘Variables Explained Away’, *Proc. of the American Philosophical Association* **140**, 343–347.