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Some challenges for compositional quantification in logic

Abstract. Facing the problem of compositionality in Tarskian implementations of quantificational logic, we propose that variables should not be interpreted as potentially referential expression, but rather "contextually" in that sequences of variables denote information about whether or not arguments of a predicate necessitate identical referential interpretations. As a result of this shift of perspective, we define a quantificational logic that is both compositional and alphabetically innocent, as discussed in Klein and Sternefeld (2017). However, classical compositional predicate logic (CPL) and alphabetically innocent predicate logic (AIL) have in common that the denotations of conjunctions like $(P(x) \wedge Q(y))$ and $(Q(y) \wedge P(x))$ differ. In CPL, this is due to the fact that the sequences $\langle x, y \rangle$ and $\langle y, x \rangle$ must be distinguished in semantic interpretation; in AIL, these sequences have the same denotations, but still the denotations of the conjuction $\langle (P \wedge Q), \langle x, y \rangle \rangle$ differs from that of $\langle (Q \wedge P), \langle x, y \rangle \rangle$ and $\langle (Q \wedge P), \langle y, x \rangle \rangle$. In this article, I will demonstrate how this asymmetry can be overcome in a system that replaces sequences of variables with sets of thematic roles.

Keywords: Tarskian quantification, truth, compositional predicate logic, alphabetic innocence, asymmetric conjunction, local symmetry

1. Introduction

It's widely acknowledged that the conventional implementation of Tarskian quantificational logic lacks compositionality. This shortcoming stems from its reliance on assignment functions to handle variables. Consequently, the denotation of $[\![\forall x \phi]\!]^g$ doesn't solely hinge on $[\![\phi]\!]^g$, but instead depends on alternatives to the assignment function g. Nevertheless, this approach seems grounded in truth, as the denotation of $[\![\phi]\!]^g$ equates to a truth value.

In their essay titled "Against Fregean Quantification", Pickel and Rabern (2022) demonstrate that the Fregean implementation of quantification also

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lacks compositionality. Despite the seemingly compositional nature of its fundamental elements – where quantifiers manifest as second-order properties and sentences as truth value bearers – the Fregean semantics leans on a mechanism to construct complex predicates. This mechanism, often represented in modern semantics by lambda expressions like $\lambda x(P(x) \wedge Q(x))$, can be shown to be non-compositional, much as $\forall x$ is in the Tarskian framework. But then, neither framework complies with the very idea of compositionality.

In contrast, the aforementioned authors explore an alternative Tarskian-style interpretation that indeed upholds compositionality but diverges from their principle of TRUTH CENTRALITY (p. 975):

(1) The denotation of a sentence is a possible argument for a truth function. Therefore, the denotation of a sentence is its truth value.

Their focal point of concern revolves around the treatment of (open) propositions as sets of assignment functions. For instance, the proposition $P(x,y,z)$ is transformed into the formula $\lambda g.P(g(x),g(y),g(z))$. It's evident that these denotations do not correspond to truth values. Nonetheless they lend themselves to a compositional semantics of quantification whose building blocks are sets of assignment functions. As a result, then, it seems that a truely compositional semantics is necessarily at odds with TRUTH CENTRALITY.

Furthermore, Pickel and Rabern (2022) raise the legitimate concern that denotations based on assignment functions lack alphabetical invariance. This means that the denotions of $P(x)$ and $P(y)$ are not equivalent, even in the compositional variant of Tarskian quantificational logic. Lack of invariance is obvious in the traditional frameworks, with their interpretations residing on assignment functions. However, the same still holds when we switch to denotations as *sets* of assignment functions, where $P(x)$ and $P(y)$ are interpreted as $\lambda g.P(g(x))$ and $\lambda g.P(g(y))$ respectively. Hence lack of alphabetic invariance seems unavoidable. This is what Kit Fine (2007) has called the "antinomy of the variable".

2. Discussion

Having sketched these authors' dilemmas, I'd like to articulate my own reservations. From my perspective, once we introduce variables into our theoretical framework (unlike Quine (1960), who dismisses with variables altogether), it becomes unreasonable to embrace the idea of truth centrality as one side of a dilemma. It even appears counterintuitive that open propositions should inherently denote truth values. For Pickel and Rabern, the strategic importance of (1) appears to stem from its potential validity within Fregean semantics, a framework that starts with sentences rather than open propositions. However, upon realizing that this step doesn't offer much when sentences need to be transformed into expressions with Fregean "gaps" that resist a straightforward compositional interpretation, it seems to me that this principle has essentially lost its credibility.

As a result, the centrality of truth may still be a trivial concern, particularly when our ultimate goal is to define truth conditions. However, this doesn't necessarily extend to all aspects of the language, not even to sentences. Instead, a more reasonable and achievable objective — one that can effectively serve as a tool for formulating truth conditions -is to establish a coherent and reasonable notion of *satisfiability*. Of course, one could adhere to the idea that sentences denote nothing other than truth values. The issue, however, lies in the coordination between a sentence and an open formula. Their semantics would lack uniformity, a clear weakness that could be avoided by focusing solely on satisfiability.

Turning next to alphabetical innocence, the point seems to be well taken. However, upon further reflection it transpires that lack of innocence is indeed unavoidable once we acknowledge that variables themselves are part of denotations. And this is precisely the situation that arises with assignments: the central drawback of compositional assignment semantics lies in the fact that assignments (as building blocks and components of denotations proper) are functions from variables to referents. Consequently, variables themselves become integral elements of the theory's ontology.

Indeed, the most unsettling consequence is not that truth is no more "central" in the sense of (1), but lies in the fact that our commitment to assignments implies an ontological commitment to variables as semantic objects. This, unfortunately, inevitably leads to alphabetical non-innocence and a blurring of the distinction between object-language and meta-language. This dual role of variables, both as elements in the language of logic and as objects manipulated by assignment functions, creates a situation where they must be interpreted as names for themselves. This blurring of boundaries between language and ontology is well articulated in Chapter 10 of Zimmermann and Sternefeld (2013), as also referenced in footnote 13 of Pickel and Rabern (2022). Zimmermann and Sternefeld thus conclude that such language-dependent extensions are deceptive (they are "somewhat of a cheat", p. 243). The core dilemma then appears to be that we are forced to either admit questionable (language-like) entities into our ontology or relinquish the pursuit of compositionality.

3. Compositionality and alphabetic innocence regained

A potential resolution of this dilemma involves a redefinition of the role of variables in semantic composition. Traditionally, variables are perceived as contributing referentially, serving as placeholders for specific objects. However, we propose relinquishing the notion that referentiality is their central function. Instead, consider interpreting an expression of the form $P(x_1,\ldots,x_n)$ in a manner where variables simply indicate which positions of P should receive interpretations that are either identical or potentially distinct from each other.

Given that part of an open proposition is a sequence of variables $\langle x_1,\ldots,x_n\rangle$, we can interpret this sequence as implying that the positions of P must share the same interpretation only if the symbol referred to by our metalinguistic variable x_i is identical to the symbol referred to by x_j. In the realm of ordinary semantics, our ontological commitment pertains to sequences of objects. However, under the present framework, the new role of variables is to merely give an instruction to interpret potential objects of a relation as identical given the corresponding positions of the predicate are identical in the sequence that encodes these positions.

This approach ensures that the denotation of a sequence like *xyz* is not only equivalent but identical to that of ▽○□ (three distinct positions), and that of *xyx* is the same as that of *yxy* or of □○□ or ○□○. This guarantees alphabetic innocence.

Any such sequence of symbols will be called a σ-sequence. Formally, there are various way to represent the denotation of a σ-sequence; these can *e.g.* be encoded by equivalence classes of numbers that refer to the positions of the sequence, or by sets $\langle n,m,\nu\rangle$ where n and m are numbers and ν specifies whether the positions at n and m must have identical denotations. A rigorous formalization of this concept can be found in Klein and Sternefeld (2017), executed in precise detail.

However, for our current purposes, we will adopt a more informal approach and identify the denotation of an n-place sequence σ directly with the largest n-place relation over a domain D that complies with the condition in (2) (here, π_i refers to the i-th projection of a sequence, *i.e.* the i-th element of an n-tuple):

(2) $[\![\sigma]\!]_D = \{s : s \in D^n \text{ and if } \pi_i(\sigma) = \pi_j(\sigma), \text{ then } \pi_i(s) = \pi_j(s)\}$.

In order to see the relevance of this notion, consider the following definitions of satisfiability, truth, and falsity in (3):

(3) Assume that P denotes an n-place relation $[\![P]\!]$, and let σ be an n-place sequence of symbols, also called variables. Then

a. $\langle P, \sigma \rangle$ is satisfiable if $[\![P]\!] \cap [\![\sigma]\!]_D \neq \emptyset$

b. $\langle P, \sigma \rangle$ is false if $[\![P]\!] \cap [\![\sigma]\!]_D = \emptyset$

c. $\langle P, \sigma \rangle$ is true if $[\![\sigma]\!]_D \subseteq [\![P]\!]$

E.g., let R denote $\{\langle a,b \rangle, \langle b,a \rangle\}$ and $\sigma = \Box\Box$. Then $[\![\sigma]\!]_D = \{\langle a,a \rangle, \langle b,b \rangle, \langle c,c \rangle, \ldots\}$ and $\langle R, \sigma \rangle$ is false. On the other hand, if R denotes identity, then $[\![R]\!] = [\![\sigma]\!]_D$ is true (and satisfiable). If $\sigma = \Box\bigcirc$, then $[\![\sigma]\!]_D = D^2$ and $\langle R, \sigma \rangle$ is satisfiable but not true.

Our next step is to define a formal language of predicate logic, in tandem with its semantic interpretation. For ease of exposition we will not include names for individuals in our version of predicate logic. For simple predicate constants, clause (4-a) states the relevant conditions, as does (4-b) for negation:

(4) a. Let P be a predicate constant of arity n, I the interpretation function (cf. (8-b)) of a model $M = \langle I, D \rangle$ with domain D, and σ a sequence of symbols of length n. Then

(i) $\langle P, \sigma \rangle \in Fml_n$ and

(ii) $[\![\langle P, \sigma \rangle]\!] = \langle I(P), [\![\sigma]\!]_D \rangle$.

b. Let $\langle \alpha, \sigma \rangle \in Fml_n$. Then

(i) $\langle \neg\alpha, \sigma \rangle \in Fml_n$ and

(ii) $[\![\langle \neg\alpha, \sigma \rangle]\!] = \langle D^n \setminus [\![\alpha]\!], [\![\sigma]\!]_D \rangle$.

These clauses should be obvious, at least if $n \neq 0$. But what about closed formulas, *i.e.* those that do not contain free variables? As will be shown below, (non-vacuous) quantification reduces the arity of a relation. A formula without free variables is an element of Fml_0, with σ being the empty sequence $\langle\rangle$. Assuming that the i-th projection of $\langle\rangle$ is undefined (or $\emptyset$), the condition in (2) is vacuously met, hence $[\![\langle\rangle]\!]_D = \{s : s \in D^0\}$. Since $D^0 = \{\langle\rangle\}$, $s = \langle\rangle$, therefore $[\![\langle\rangle]\!]_D = \{\langle\rangle\} = \{\{\}\} = \{\emptyset\}$.

As concerns the first element ϕ of a formula $\langle \phi, \sigma \rangle$ with arity zero, two possibilities arise (cf. (4-d) below): either the conditions that reduce ϕ's arity from 1 to 0 can be satisfied, then $[\![\phi]\!] = D^0$. Or the conditions cannot be met, then $[\![\phi]\!] = \emptyset$. Identifying truth with $\{\emptyset\}$ and falsity with $\emptyset$, (3) implies that a closed formula $[\![\phi, \langle\rangle]\!]$ is false if $[\![\phi]\!] = \emptyset$ and it is true if $[\![\phi]\!] = \{\emptyset\}$.

Returning to (4-b), recall that $D^0 \setminus \{\emptyset\} = \emptyset$ and $D^0 \setminus \emptyset = D^0$; it thus follows that negation of a closed formula maps truth onto falsity and vice versa.

As for conjunction, we have to solve a well-known problem. Suppose ϕ is $\langle P, \sigma\rangle$ and $\sigma = \langle xyz\rangle$ and we want to conjoin ϕ with a one-place predicate ψ, *e.g.* with $\langle Q, \Box\rangle$. In ordinary predicate logic, we automatically make a choice as to whether $\Box$ is one of x, y, z, or a new variable v. Given alphabetic innocence, there is no such notational automatism; rather we explicitly have to state what Fine (2007) has called a *coordination scheme*. In our implementation, this is simply a new sequence σ' such that the first three elements of σ' are isomorphic to $\langle x, y, z\rangle$ and the forth element is defining the coordination by either being new or by being identical to one of the three previous symbols.

In general, if σ is n-place, σ' is m-place, we define $\sigma @ \sigma'$ as the set of $n+m$-place sequences $\langle \tau\tau'\rangle$ such that τ and τ' are alphabetic variants of σ and σ' respectively. Coordination then requires to specify an element of $\sigma @ \sigma'$ and the Cartesian product $\otimes$ of relations:

(4) c. Let $\langle \alpha, \sigma_n\rangle \in Fml_n$, $\langle \beta, \sigma_m\rangle \in Fml_m$, and $\kappa \in \sigma_n @ \sigma_m$. Then
   (i) $\langle(\alpha \wedge \beta), \kappa\rangle \in Fml_{n+m}$ and
   (ii) $[\![\langle(\alpha \wedge \beta), \kappa\rangle]\!] = \langle [\![\alpha]\!] \otimes [\![\beta]\!], [\![\kappa]\!]_D\rangle$.

Turning finally to quantification, it should be abvious that the identification of the variable to be bound also requires a kind of coordination scheme. Suppose again that $\phi = \langle P, \sigma\rangle$ and $\sigma = \langle xyz\rangle$. In ordinary predicate logic, expressions like $\exists y$ when prefixed to ϕ, automatically target the second variable of ϕ for quantification. In our implementation, this choice can be implemented by another κ that is isomorphic to $\langle y, x, y, z\rangle$ whose first element identifies the target of the quantification. Given this sequence, we can then form a reduced sequence by removing from $\langle y, x, y, z\rangle$ the first and the third element, *i.e.* all elements that are identical with the first. Let r_1 denote the function that strips off all elements identical to the first. $r_1(\langle y, x, y, z\rangle) = \langle x, z\rangle$.

The semantics works in a parallel fashion: given a formula $\langle \phi, \sigma\rangle$, we first see to it that ϕ conforms to σ by forming $[\![\phi]\!] \cap [\![\sigma]\!]$. We then form an $n+1$ place relation $R = D \otimes ([\![\phi]\!] \cap [\![\sigma]\!])$ and finally cut away the first projection of R and all projections that are targets of the first. We refer to the result as to $r_1(\langle \phi, \kappa\rangle)$. To illustrate, let $[\![\alpha]\!] = \{\langle a, b\rangle, \langle b, b\rangle\}$ and $\sigma = \Box\bigcirc$. Then $[\![\sigma]\!]_D = D^2$ and $[\![\alpha]\!] \cap [\![\sigma]\!]_D = [\![\alpha]\!]$. We then form $D \otimes [\![\alpha]\!]$, a three place relation. Let $\kappa = \langle xyx\rangle$. Performing the respective cut by eliminating the positions corresponding to x yields $r_1(\langle \alpha, \langle xyx\rangle) = \{\langle a\rangle, \langle b\rangle\}$ and $r_1(\langle xyx\rangle) = \langle y\rangle$.[1]

[1] This and all other operations in this section were recursively defined in Klein and Sterne-

The general definition now reads as follows:

(4) d. Let $\phi = \langle \alpha, \sigma \rangle \in Fml_n$, k a symbol, and $\kappa \in k@\sigma$. Then
(i) $\langle \exists\kappa\phi, r_1(\kappa) \rangle \in Fml_m$ where m is the arity of $r_1(\kappa)$ and
(ii) $[\![\langle \exists\kappa\phi, r_1(\kappa)\rangle]\!] = \langle r_1(\langle D \otimes ([\![\alpha]\!] \cap [\![\sigma]\!]_D), \kappa\rangle), [\![r_1(\kappa)]\!]_D\rangle$.

To illustrate, let us calculate the analogue of $\exists x R(x,x)$, assuming that $[\![R]\!] = \{\langle a,b\rangle, \langle a,c\rangle, \langle a,a\rangle\}$. Intersection with $[\![xx]\!]_D$ (or $[\![\equiv]\!]$) yields $\{\langle a,a\rangle\}$ and $D \otimes \{\langle a,a\rangle\} = \{\langle a,a,a\rangle, \langle b,a,a\rangle, \ldots\}$. Since x is targeted, all positions must go, resulting in the zero-place relation $\{\emptyset\}$. Intersection with the empty sequence $\{\emptyset\}$ from (2) yields $\{\emptyset\}$ and the formula is true. Suppose $[\![R]\!] = \{\langle a,b\rangle, \langle a,c\rangle\}$. Intersection with identity yields $\emptyset$. $D \otimes \emptyset = \emptyset$ and $\emptyset \cap \{\emptyset\} = \emptyset$, hence the formula is false.

One of the reviewers suggests that the reader would benefit from “a concrete theorem on the equivalence of the new logic to classical first order logic”. I will briefly review some of the formal results first published in Klein and Sternefeld (2017). All proofs can be found there; here, I will only sketch the main ideas.

First note that there is a straightforward way to translate from CPL to AIL and vice versa. Since alphabetic invariance introduces indeterminacy, we have to introduce some desambiguating conventions; without loss of generality both languages should agree on their use of variables. It is clear, then, that $P(x,y,z)$ unambiguously translates as $\langle P, \langle xyz\rangle\rangle$ (written as $T(P(x,y,z))$, and vice versa. Given the identity of variables, it is easy to define translations by induction on the complexity of expressions of both languages.

Second, we observe a correspondance between the interpretation of variables. In compositional logic, assignment functions interpret variables, and by the coincidence theorem it is clear that we can reduce these functions to the parts that interpret the free variables of a formula. These parts then directly conform to the sequences in $[\![\sigma]\!]_D$. We can define this correspondence as in (5-b):

(5) a. $g(\langle x_1, x_2, \ldots, x_n\rangle) := \langle g(x_1), g(x_2), \ldots, g(x_n)\rangle$.
b. $GtoS(A) := \{s : \text{there is a } g \in [\![A]\!] \text{ such that } s = g(\pi_2(T(A)))\}$.

(6) Lemma: $GtoS(A) = \pi_1([\![T(A)]\!]) \cap \pi_2([\![T(A)]\!])$

The proof procedes by induction over the complexity of formulas.

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feld (2017) by induction on the length of expressions. In this paper I prefer an informal presentation; the required definitions can be adopted from there.

Finally, it can be shown that A is satisfiable (true/false) iff $T(A)$ is satisfiable (true/false). Consider satisfiability:

(7) Proof: A is satisfiable iff $[\![A]\!] \neq \emptyset$ iff $GtoS(A) \neq \emptyset$ iff $\pi_1([\![T(A)]\!]) \cap \pi_2([\![T(A)]\!]) \neq \emptyset$ iff $T(A)$ is satisfiable.

See Klein and Sternefeld (2017) for further discussion.

4. Local symmetry

Having achieved alphabetic innocence and compositionality, the new implementation AIL still shares an objectionable property with CPL. As a consequence of relying on sequences of variables and on denoatations as sequences of individuals, the extensions of $(P(x) \wedge Q(y))$ and $(Q(y) \wedge P(x))$ differ, and so do the denotations $P \otimes Q$ and $Q \otimes P$ in the present system. This has been discussed as the problem of *local asymmetry* in Klein and Sternefeld (2017). In this paper, I will discuss a new solution to the problem.[2]

Clearly, the problem of local asymmetry is related to the fact that the syntax of simple predicates already determines a certain ordering of arguments that is mapped into its semantics. As for complex predicates (generated by conjunction), this ordering is simply inherited by the ordering of the conjuncts. From a linguistic point of view, this part of the implementation of predicate logic is problematic in as far as the order of arguments of a predicate, viewed as the translation of an expression of natural language, could be language specific and arguably should not be encoded in its semantics. A pure semantics for natural language should therefore dispense with a pregiven order, the latter being a matter of syntax alone (cf. section 4.5. in Zimmermann and Sternefeld (2013)). But then, with the ordering of arguments being of no avail in pure semantics, $(P \wedge Q)$ and $(Q \wedge P)$ should also denote the same objects.

In linguistics, a standard order free semantics is embodied in the theory of states and events: Each predicate P comes along with a state or event e and a set of n different thematic roles θ_i that specify a certain thematic relation (agent, patient, beneficient, etc.) between an individual and a state or event (*cf.* Parsons (1990)). Part of the theory (called *linking theory*) maps thematic relations onto

[2] A different, but minor problem might be seen in the compositionality of the coordination schemes. By using methods discussed in Quine (1960), sequences of symbols, although potentially infinite, can be defined recursively, as can their semantics. This is not directly expressed by the semantic rules stated above, but in principle, this could be added in order to make for a fully compositional semantics for all operations on σ and their denotations.

a linear order that mirrors the order of arguments in natural language (and, by analogy, in standard predicate logic). However, such a theory is incompatible with standard predicate logic. In particular, thematic roles are understood as so-called role types which represent semantically contentful relations between events/states and individuals. There are no such relations in ordinary predicate logic.

However, it might still be possible to dismiss with states and events and with a genuine semantics for roles. Rather, we could make use of what has been called *individual* thematic roles", as apposed to "thematic role *types*". *E.g.* it has been argued that an object of *like* is neither a theme nor a patient (or any other run of the mill role types), but rather a "likee", which is an individual thematic role of *like*. To a certain extent, this line of reasoning reflects a well-known difficulty for theorists of thematic relations to properly identify role types.

Instead, individual roles merely serve as symbols that identify arguments without explicitly bringing them into a syntactic order. The following schematic comparison shows how an ordinary predicate extension relates to a thematic one (taken from Zimmermann and Sternefeld (2013) p. 80):

(8) a. Ordinary: $I(P^n) = \{\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots y_n\rangle, \ldots\}$
b. Thematic: $I(P^n) = \{\{\langle \theta_1, x_1\rangle, \ldots \langle \theta_n, x_n\rangle\}, \{\langle \theta_1, y_1\rangle, \ldots \langle \theta_n, y_n\rangle\}, \ldots\}$

The sequences in (8-a) are effectively replaced by sets in (8-b). Each predicate comes along with its own set of individual thematic roles (the "biter", the "bitee", cf. Marantz (1981)). The thematic roles as such do not contain any information about the ordering of elements in (8-a). Therefore, the index n on the thematic role in (8-b) is independent from and not to be confused with the index n on the variable in (8-a). Each n-place predicate P can thus be identified with its set of (pairwise distinct) thematic roles $\{\theta_1^P, \ldots, \theta_n^P\}$ whose interpretation I in a model is as indicated in (8-b). *E.g.*, R = *beat* and R^- = *being-beaten* are different predicates with different thematic roles, although of course a *beatee* is necessarily a *being-beaten*.[3]

Let us first define the restrictions on a relation imposed by identity of positions. We will account for former equality of variables directly by statements about identity of (values for) thematic roles:

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[3] As a consequence, two predicates may be synonymous without having the same denotations. This is because theta roles are "privative" (*i.e.* specific for each predicate) – a matter to which I return in Section 5.

(9) Let σ be a (possibly empty) set of formulas of the form $\theta_i \equiv \theta_j$, $1 \leq i < j \leq n$. We define $[\![\langle n, \sigma\rangle]\!]_D$ as the set of all (unordered) n-place relations that satisfy the restrictive statements in σ. Moreover, let $[\![\langle 0, \emptyset\rangle]\!]_D = \{\emptyset\}$.

Although the characterization of $[\![\langle n, \sigma\rangle]\!]_D$ is a bit informal, it should be clear that its semantics is basically the same as that of σ in our previous account, namely the set of relations that conform to certain restrictions regarding the identity of participants of the predicate encoded as thematic roles. More explicitly, the conditions hold for a relation R iff for all $\alpha \in R$, if $\langle \theta_i, a\rangle$ and $\theta_j, b \in \alpha$, then $a = b$. It should also be obvious that the former definitions of truth, falsity, and satisfiability carry over to the new format of relations.

We are now in a position to state the interpretation of atomic formulas, negation, and conjunction. As usual, negation is a function that maps an n-place relation onto its n-place complement. Conjunction forms a complex predicate with theta roles that pertain to both of its conjuncts, hence the use of different superscripts in (10-b-ii). The mode of combination of the denotation sets of predicates is analogous to the formation of Cartesian products of sequences.

(10) a. Let $P = \{\theta_1^P, \ldots, \theta_n^P\}$ be an n-place predicate, σ a (possibly empty) set of formulas of the form $\theta_i^P \equiv \theta_j^P$, $i \neq j$, and $\theta_i^P, \theta_j^P \in P$. Then
(i) $\langle P, \langle n, \sigma\rangle\rangle \in Fml_n$ and
(ii) $[\![\langle P, \langle n, \sigma\rangle\rangle]\!] = \langle I(P), [\![\langle n, \sigma\rangle]\!]_D\rangle$.

b. Let $\langle \alpha, \langle n, \sigma\rangle\rangle \in Fml_n$. Then
(i) $\langle \neg\alpha, \langle n, \sigma\rangle\rangle \in Fml_n$ and
(ii) $[\![\langle \neg\alpha, \langle n, \sigma\rangle\rangle]\!] = \langle [\![\neg\alpha]\!], [\![n, \sigma]\!]_D\rangle$, where $[\![\neg\alpha]\!] = \{s : s = \{\langle \theta_1^\gamma, x_1\rangle, \ldots, \langle \theta_n^\beta, x_n\rangle\}$ such that $\theta_1^\gamma, \ldots \theta_n^\beta \in \alpha$ and $s \notin [\![\alpha]\!]\}$.

c. Let $\langle \alpha, \langle n, \sigma_1\rangle\rangle \in Fml_n$, $\langle \beta, \langle m, \sigma_2\rangle\rangle \in Fml_m$, and σ_3 a (possibly empty) set of formulas of the form $\theta_i^\alpha \equiv \theta_j^\beta$ with $\theta_i^\alpha \in \alpha$ and $\theta_j^\beta \in \beta$. Then
(i) $\langle (\alpha \wedge \beta), \langle n+m, \sigma_1 \cup \sigma_2 \cup \sigma_3\rangle\rangle \in Fml_{n+m}$ and
(ii) $[\![\langle (\alpha \wedge \beta), \langle n+m, \sigma_1 \cup \sigma_2 \cup \sigma_3\rangle\rangle]\!] = \langle \bigcap\{\Gamma :$ if $a \in [\![\alpha]\!]$ and $b \in [\![\beta]\!]$, then $a \cup b \in \Gamma\}, [\![\langle n+m, \sigma_1 \cup \sigma_2 \cup \sigma_3]\!]_D\rangle\rangle$.

Clearly, σ_3 of conjunction functions as a coordination scheme. Note also that $[\![\langle n+m, \sigma_1 \cup \sigma_2 \cup \sigma_3\rangle\rangle]\!]_D = [\![\langle n+m, \sigma_1\rangle]\!]_D \cap [\![\langle n+m, \sigma_2\rangle]\!]_D \cap [\![\langle n+m, \sigma_3\rangle]\!]_D$.

It remains to account for quantification. Suppose, some θ_i is the target of quantification and suppose σ contains $\theta_i \equiv \theta_j$. Then θ_j should also be a target and we also want θ_k to be a target in case $\theta_j \equiv \theta_k \in \sigma$. We thus want to consider a kind of transitive closure under identity for all identity statements in σ. Let τ be such a set of thematic roles, the target positions generated by θ_i and σ.[4]. If τ has m elements, m theta roles must be paired with the same argument and quantification of an n-place relation over these arguments will yield a new relation of arity $n-m$ in which all targets have been removed. This is stated more explicitly in (10-d.):

(10) d. Let $\phi = \langle \alpha, \langle n, \sigma \rangle \rangle \in Fml_n$ and τ a set of m targets. Then
   (i) $\langle \langle \exists \tau, \phi \rangle, \langle n-m, \sigma' \rangle \rangle \in Fml_{n-m}$, where σ' results from σ by removing all identity statements that involve targets in τ.
   (ii) $[\![\langle \langle \exists \tau, \phi \rangle, \langle n-m, \sigma' \rangle \rangle]\!] = \langle \{ s :$ for some $s' \in [\![\alpha]\!] \cap [\![\langle n, \sigma \rangle]\!]$, s results from s' by deleting in s' all elements $\langle \theta_i, x_j \rangle$ where $\theta_i \in \tau \}, [\![\langle n-m, \sigma' \rangle]\!] \rangle$.

To demonstrate, consider an analogue of $\exists y Q(x,y,x,y)$ in (11):[5]

(11) $\langle \langle \exists \{ \theta^Q_2, \theta^Q_4 \}, \langle \{ \theta^Q_1, \ldots \theta^Q_4 \}, \langle 4, \{ \theta^Q_1 \equiv \theta^Q_3, \theta^Q_2 \equiv \theta^Q_4 \} \} \rangle, \langle 2, \{ \theta^Q_1 \equiv \theta^Q_3 \} \rangle \rangle \in Fml_2$

Similarly, a closed formula like $\exists x R(x,x)$ can be represented as (12):

(12) $\langle \langle \exists \{ \theta^R_1 \theta^R_2 \}, \langle \{ \theta^R_1, \theta^R_2 \}, \langle 2, \{ \theta^R_1 \equiv \theta^R_2 \} \rangle, \langle 0, \emptyset \rangle \rangle \in Fml_0$

Looking at the semantic interpretation of (12), assume as in section 3 that $[\![R]\!] = \{ \langle a,b \rangle, \langle b,a \rangle \}$. Then (13-a) is an interpretations in CLP, that in (13-b) is the corresponding one in AIL and (13-c) the one in SAIL:

(13) a. $[\![R(x,x)]\!] = \{ g : \langle g(x), g(x) \rangle \in [\![R]\!] = \emptyset$
   b. $[\![\langle R, \langle x,x \rangle \rangle]\!] = [\![R]\!] \cap [\![\langle xx \rangle]\!] = [\![R]\!] \cap Id_= = \emptyset$
   c. $[\![\langle \{ \theta^R_1, \theta^R_2 \}, \langle 2, \{ \theta^R_1 \equiv \theta^R_2 \} \rangle \rangle]\!] = \{ \{ \langle \theta^R_1, a \rangle, \langle \theta^R_2, b \rangle \}, \{ \langle \theta^R_1, b \rangle, \langle \theta^R_2, a \rangle \} \} \cap \{ Q : \{ \langle \theta^Q_1, x \rangle, \langle \theta^Q_2, x \rangle \} \in Q \} = \emptyset$

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[4] Technically, τ could be empty or contain some $\theta_j \notin \alpha$. This would amount to vacuous quantification. If not, then for some $\theta_i \in \alpha$ (the target), τ is generated by θ_i iff τ is the smallest set such that (a) $\theta_i \in \tau$, (b) if $\theta_i \equiv \theta_j$ or $\theta_j \equiv \theta_i \in \sigma$, then $\theta_j \in \tau$, and (c) if $\theta_k \in \tau$ and $\theta_k \equiv \theta_l$ or $\theta_l \equiv \theta_k \in \sigma$, then $\theta_l \in \tau$

[5] Note that, in principle, the formula in (11) could also be an analogue of $\exists y Q(y,y,x,x)$, $\exists y Q(y,x,x,y)$, $\exists x Q(y,y,x,x)$ etc., whatever convention is used to pair positions with theta-roles θ_i.

As the intersection in (13-c) is empty, there is no such s' as required in (10-d-ii) and existential quantification is false. On the other hand, if R contains $\{\langle\theta_1^R, a\rangle, \langle\theta_2^R, a\rangle\}$, such an s' exists, but deleting the targeted theta positions leaves us with $\{\emptyset\}$. Intersection with $[\![\langle 0, \emptyset\rangle]\!]_D = \{\emptyset\}$ yields $\{\emptyset\}$, hence the statement is true.

In general, there cannot be a one-to-one correspondance between formulas of AIL and SAIL, unless we specify an ad-hoc ordering of all theta-roles. This odering directly maps sets onto sequences, hence the semantic objects of SAIL directly correspond to those of AIL. Note also that the translation of a "symmetric" formula of SAIL is no more ambiguous, as it will depend on the ordering of the theta roles involved, which in turn induces an ordering of the predicates. It therefore comes as no surprise that the truth conditions of an "ordered" SAIL are the same as that of AIL.

5. A final remark

To sum up, truth cannot serve as the focal mechanism for compositional predicate logic; furthermore, assignment functions are problematic for ontological reasons. Therefore, a relational system emerges as a viable alternative, this being in addition alphabetically innocent. Nevertheless, the persistence of local asymmetry remains as an undesirable feature of the system.

As a potential resolution, we have devised a semantic system grounded in individual thematic roles. However, it's crucial to acknowledge that thematic roles, whether categorized as role types or individual roles, are somewhat dubious entities at best. In our earlier discussion, we lauded the present framework for its virtue in rendering thematic roles "contentless" (*i.e.* they do not signify relations between individuals and events or states). Therefore, referring to them as "thematic" can be largely misleading. Instead, their primary function lies in identifying a lexical item and its set of argument slots, rather than determining its grammatical function. Consequently, the SAIL-system as such is not designed to align with morpho-syntactic properties of natural language.

However, this does not imply that SAIL is incompatable with a contentful system of role types. If Dowty (1991) is correct, thematic role types, which are meant to determine linear orderings and grammatical functions, are not primitive entities, but are instead defined in terms of entailments between propositions. As a consequence, these role types can also be defined in SAIL. *E.g.*, if in all models the entitiy referred to as a "beater" also has the property of being "animated" and an "actor" (*i.e.* the proposition *John beats Mary* entails *John* *is animated*, etc.), the “beater” will qualify as the subject of *beat*. While this representation is admittedly a simplified outline of Dowty’s theory, it brings positive implications for SAIL. The fact that entailment relations remain consistent irrespective of the nomenclature assigned to theta roles means there is no necessity to delve into the speculative nature of what they may “truly” represent. Given that the entailment relations hold regardless of how theta roles are named, there is no need to bother about what they “really” are.

On the flip side, theta roles form an integral part of SAIL’s ontology, and being individual theta roles they inherently contain an implicit reference to the predicate itself. In a sense, this does not come as a surprise. For compare SAIL with the conventional implementation of theta roles as relations between individuals and events. Condider an unanalysed individual role like *beater* in SAIL. In event theory, this is resolved into a relation “-er” between an individual and an event, namely the event of beating. In consequence, theta theory typically involves an ontological commitment to events that pertain to the predicates whose meaning is at stake. It appears, therefore, that the specific impact of an individual theta role (incorporating the predicate to which it pertains) directly aligns with that of the corresponding event in event theory.

At the end of the day, there is no free dinner: as soon as we dispense with linear order, addition ontological commitments seem unavoidable. From my perspective, commitment to states or events appears to be a more straightforward proposal than commitment to individual theta roles. I suspect that both systems are formally equicalent. However, this could be a subject open to debate.

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