Some challenges for compositional quantification in logic

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Abstract

Facing the problem of (non-)compositionality in Tarskian implementations of quantificational predicate logic (PL), we propose that compositionality can be regained if variables are not be interpreted as potentially referential expression, but rather "contextually" in that sequences of variables merely denote information about whether or not arguments of a predicate necessitate identical referential interpretations. This shift of perspective enables us to define a quantificational logic (C+I-PL) that is both compositional (C) and alphabetically innocent (I), as discussed in Klein and Sternefeld (2017). However, an open problem addressed there is that the denotations of conjunctions like $(P(x) \wedge Q(y))$ and $(Q(y) \wedge P(x))$ differ. In a Tarskian PL, this is due to the fact that the sequences $\langle x, y\rangle$ and $\langle y, x\rangle$ must be distinguished in semantic interpretation; in C+I-PL, these sequences have the same denotations, but still the denotations of conjuctions remain asymmetric. In this article, I will demonstrate how this asymmetry can be overcome in a symmetric system C+S-PL that replaces sequences of variables with sets of thematic roles. As it will turn out, however, any such system is necessarily at odds with alphabetic innocence.

Keywords

Tarskian quantification, truth, compositional predicate logic, alphabetic innocence, local symmetry, thematic roles, linear order of arguments

1 Introduction

It's widely acknowledged that the conventional implementation of Tarskian quantificational logic lacks compositionality. This shortcoming stems from its reliance on assignment functions to handle variables. Given that the denotation function $[\![.]\!]$ depends in addition on an assignment function g, the denotation of $[\![\forall x\phi]\!]^g$ doesn't solely hinge on $[\![\phi]\!]^g$ (as one would expect in a compositional system), but instead depends on alternatives g' to g. Nevertheless, this approach seems grounded in truth, as the denotations of $[\![\forall x\phi]\!]^g$ and $[\![\phi]\!]^{g'}$ are truth values.

As an alternative to a Tarskian implementation employing assignment functions, Pickel and Rabern (2022) discuss the Fregean implementation, which hinges on quantifiers as second-order properties and sentences as truth value bearers. While this approach appears to be rooted in truth, it also suffers from a lack of compositionality. This is due to Frege's mechanism for constructing complex predicates. In modern semantics, this would be symbolized by lambda expressions like $\lambda x(P(x) \wedge Q(x))$, and it is obvious that the semantic interpretation of λx is non-compositional, as is $\forall x$ in the Tarskian framework.

Next, Pickel and Rabern (2022) discuss an alternative implementation that indeed upholds compositionality but diverges from their principle of TRUTH CENTRALITY (p. 975):

(1) The denotation of a sentence is a possible argument for a truth function. Therefore, the denotation of a sentence is its truth value.

Their focal point of concern revolves around the treatment of (open) propositions as sets of assignment functions. For instance, the proposition $P(x, y, z)$ is transformed into the formula $\lambda g.P(g(x), g(y), g(z))$. It's evident that its denotation does not correspond to a truth value. Nonetheless these denotations lend themselves to a compositional semantics C-PL of quantification, whose building blocks are sets of assignment functions. As a result, then, it seems that compositionality is necessarily at odds with TRUTH CENTRALITY.

Furthermore, Pickel and Rabern (2022) raise the legitimate concern that denotations based on assignment functions lack alphabetical invariance. This means that the denotations of $P(x)$ and $P(y)$ are not equivalent, even in C-PL. Lack of invariance is obvious in the traditional implementations of PL, with their interpretations residing on assignment functions. However, the same still holds when we switch to denotations as *sets* of assignment functions, where $P(x)$ and $P(y)$ are interpreted as $\lambda g.P(g(x))$ and $\lambda g.P(g(y))$ respectively. Hence lack of alphabetic invariance seems unavoidable. This is what Fine (2007) has called the "antinomy of the variable".

2 Discussion

In this section I'd like to articulate my own reservations against C-PL. From my perspective, once we introduce variables into our theoretical framework (unlike Quine (1960), who dismisses with variables altogether), it becomes unreasonable to embrace the idea of truth centrality as one side of a dilemma. It even appears counterintuitive that open propositions should inherently denote truth values. For Pickel and Rabern, the strategic importance of (1) appears to stem from its potential validity within Fregean semantics, a framework that starts with sentences rather than open propositions. However, upon realizing that this system, contrary to appearance, is not compositional (relying on "sentences" with "gaps"), it seems to me that this principle has essentially lost its credibility: it is highly implausible (a priori) that a compositional system can be based on truth, even if our ultimate goal is to define truth conditions for sentences.

Turning next to alphabetical innocence, the point seems to be well taken. However, upon further reflection it transpires that lack of innocence is indeed unavoidable once we acknowledge that variables themselves are part of denotations. And this is precisely the situation that arises with assignments: the central drawback of compositional assignment semantics lies in the fact that assignments (as building blocks and components of denotations proper) are functions from variables to referents. Consequently, variables themselves become integral elements of the theory's ontology.

Indeed, the most unsettling consequence is *not* that truth is no more "central" in the sense of (1), but lies in the fact that our commitment to assignments implies an ontological commitment to variables as semantic objects. This, unfortunately, inevitably leads to alphabetical non-innocence and a blurring of the distinction between object-language and meta-language. This dual role of variables, both as elements in the language of logic and as objects manipulated by assignment functions, creates a situation where they must be interpreted as names for themselves. This blurring of boundaries between language and ontology is well articulated in Chapter 10 of Zimmermann and Sternefeld (2013), as also referenced in footnote 13 of Pickel and Rabern (2022). Zimmermann and Sternefeld thus conclude that such language-dependent extensions are deceptive (they are "somewhat of a cheat", p. 243). The core dilemma then appears to be that we are forced to either admit questionable (language-like) entities into our ontology or relinquish the pursuit of compositionality.

3 Compositionality and alphabetic innocence regained

A potential resolution to this dilemma involves a redefinition of the role of variables in semantic composition. Traditionally, variables are seen as placeholders for specific objects, contributing referentially to the meaning of an expression. In contrast, we now propose relinquishing the notion that referentiality is their central function. Instead, the proposal suggests interpreting expressions in a way that focuses on the relationships between variables and their positions within an expression. For instance, in an expression of the form $P(x_1, \ldots, x_n)$ – rather than viewing $x_1, \ldots, x_n$ as placeholders for specific objects – the only contribution of variables to the meaning of the expression is to indicate which positions of P should receive interpretations that are either identical or potentially distinct from each other. Within such a framework, the new role of variables is to merely give an instruction to interpret potential arguments of a relation as identical, given the corresponding positions of the predicate are identical in the sequence that encodes these positions.

This approach ensures that the denotation of a sequence like xyz is not only equivalent but identical to that of ▽○□ (three distinct positions), and that of xyx is the same as that of yxy, or □○□, or ○▽○, etc. This guarantees alphabetic innocence.

Any such sequence of symbols will be called a σ-sequence, so that open propositions are pairs containing a relation and a σ-sequence. Formally, there are various way to represent the denotation of a σ-sequence; these can *e.g.* be encoded by equivalence classes of numbers that refer to the positions of the sequence, or by sets $\langle n, m, \nu\rangle$ where n and m are numbers and ν specifies whether the positions at n and m must have identical denotations. A rigorous formalization of this concept can be found in Klein and Sternefeld (2017), executed in precise detail.

For our current purposes, it will suffice to directly formalize the *effects* of such an identification. Towards this end, we will identify the denotation of an n-place sequence σ in terms of the predicates that comply to the restriction embodied in σ. That is, we identify the denotation if σ directly with the largest n-place relation over a domain D that satisfies the condition in (2) (here, π_i refers to the i-th projection of a sequence, *i.e.* the i-th element of an n-tuple):

(2) $[\![\sigma]\!]_D = \{s : s \in D^n \text{ and if } \pi_i(\sigma) = \pi_j(\sigma), \text{ then } \pi_i(s) = \pi_j(s)\}$

In order to see the relevance of this notion, consider the following definitions of satisfiability, truth, and falsity in (3):

(3) Assume that P denotes an n-place relation $[\![P]\!]$, and let σ be an n-place sequence of symbols, (aka variables). Then
  a. $\langle P, \sigma\rangle$ is satisfiable if $[\![P]\!] \cap [\![\sigma]\!]_D \neq \emptyset$.
  b. $\langle P, \sigma\rangle$ is false if $[\![P]\!] \cap [\![\sigma]\!]_D = \emptyset$.
  c. $\langle P, \sigma\rangle$ is true if $[\![\sigma]\!]_D \subseteq [\![P]\!]$.

To illustrate, let R denote $\{\langle a,b\rangle, \langle b,a\rangle\}$ and σ = □□. Then $[\![\sigma]\!]_D = \{\langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \ldots\}$ and $\langle R, \sigma\rangle$ is false. On the other hand, if R denotes identity, then $[\![R]\!] = [\![\sigma]\!]_D$ is true (and satisfiable). If σ = □○, then $[\![\sigma]\!]_D = D^2$ and $\langle R, \sigma\rangle$ is satisfiable but not true.

Our next step is to define a formal language of predicate logic, C+I-PL, in tandem with its semantic interpretation. For ease of exposition we will not include names for individuals in our version of predicate logic. For simple predicate constants, clause (4-a) states the relevant conditions; (4-b) defines negation as the complement of a relation:

(4) a. Let P be a predicate constant of arity n, I the interpretation function of a model $M = \langle I, D\rangle$ with domain D which maps m-place predicates into D^m, and σ a sequence of symbols of length n. Then
  (i) $\langle P, \sigma\rangle \in Fml_n$ and
  (ii) $[\![\langle P, \sigma\rangle]\!] = \langle I(P), [\![\sigma]\!]_D\rangle$.

b. Let $\langle\alpha,\sigma\rangle \in \mathit{Fml}_n$. Then
   (i) $\langle\neg\alpha,\sigma\rangle \in \mathit{Fml}_n$ and
   (ii) $[\![\langle\neg\alpha,\sigma\rangle]\!] = \langle D^n \setminus [\![\alpha]\!], [\![\sigma]\!]_D\rangle$.

These clauses should be obvious, at least if $n \neq 0$. But what about closed formulas, *i.e.* those that do not contain free variables? First consider σ. As will be shown below, (non-vacuous) quantification reduces the arity of a relation. A formula without free variables is an element of Fml_0, with σ being the empty sequence $\langle\rangle$. Assuming that the i-th projection of $\langle\rangle$ is undefined (or $\emptyset$), the condition in (2) is vacuously met, hence $[\![\langle\rangle]\!]_D = \{s : s \in D^0\}$. Since $D^0 = \{\langle\rangle\}$, $s = \langle\rangle$, therefore $[\![\langle\rangle]\!]_D = \{\langle\rangle\} = \{\{\}\} = \{\emptyset\}$.

As concerns the first element ϕ of a formula $\langle\phi,\sigma\rangle$ with arity zero, two possibilities arise: either the conditions that reduce ϕ's arity from 1 to 0 can be satisfied, then $[\![\phi]\!] = D^0$. Or the conditions cannot be met, then $[\![\phi]\!] = \emptyset$. Identifying truth with $\{\emptyset\}$ and falsity with $\emptyset$, (3) implies that a closed formula $\langle\phi,\langle\rangle\rangle$ is false if $[\![\phi]\!] = \emptyset$ and it is true if $[\![\phi]\!] = \{\emptyset\}$.

Returning to (4-b), recall that $D^0 \setminus \{\emptyset\} = \emptyset$ and $D^0 \setminus \emptyset = D^0$; it thus follows that negation of a closed formula maps truth onto falsity and vice versa.

As for conjunction, we have to solve a well-known problem. Suppose ϕ is $\langle P,\sigma\rangle$ and $\sigma = \langle xyz\rangle$ and we want to conjoin ϕ with a one-place predicate ψ, *e.g.* with $\langle Q,\Box\rangle$. In ordinary predicate logic, we automatically make a choice as to whether $\Box$ is one of x, y, z, or a new variable v. Given alphabetic innocence, there is no such notational automatism; rather we explicitly have to state what Fine (2007) has called a *coordination scheme*. In our implementation, this is simply a new sequence σ' such that the first three elements of σ' are isomorphic to $\langle x,y,z\rangle$ and the forth element is defining the coordination by either being new or by being identical to one of the three previous symbols.

In general, if σ is n-place, σ' is m-place, we define $\sigma@\sigma'$ as the set of $n+m$-place sequences $\langle\tau\tau'\rangle$ such that τ and τ' are alphabetic variants of σ and σ' respectively. Coordination then requires to specify an element of $\sigma@\sigma'$ and the Cartesian product $\otimes$ of relations:

(4) c. Let $\langle\alpha,\sigma_n\rangle \in \mathit{Fml}_n$, $\langle\beta,\sigma_m\rangle \in \mathit{Fml}_m$, and $\kappa \in \sigma_n@\sigma_m$. Then
   (i) $\langle(\alpha \wedge \beta),\kappa\rangle \in \mathit{Fml}_{n+m}$ and
   (ii) $[\![\langle(\alpha \wedge \beta),\kappa\rangle]\!] = \langle[\![\alpha]\!] \otimes [\![\beta]\!], [\![\kappa]\!]_D\rangle$.

Turning finally to quantification, it should be obvious that the identification of the variable to be bound also requires a kind of coordination scheme. Suppose again that $\phi = \langle P,\sigma\rangle$ and $\sigma = \langle xyz\rangle$. In ordinary predicate logic, expressions like $\exists y$ when prefixed to ϕ, automatically target the second variable of ϕ for quantification. In our implementation, this choice can be implemented by another κ that is isomorphic to $\langle y,x,y,z\rangle$ whose first element identifies the target of the quantification. Given this sequence, we can then form a reduced sequence by removing from $\langle y,x,y,z\rangle$ the first and the third element, *i.e.* all elements that are identical with the first. Let r_1 denote the function that strips off all elements identical to the first. $r_1(\langle y,x,y,z\rangle) = \langle x,z\rangle$.

The semantics works in a parallel fashion: given a formula $\langle\phi, \sigma\rangle$, we first see to it that ϕ conforms to σ by forming $[\![\phi]\!] \cap [\![\sigma]\!]$. We then form an $n+1$ place relation $R = D \otimes ([\![\phi]\!] \cap [\![\sigma]\!])$ and finally cut away the first projection of R and all projections that are targets of the first. We refer to the result as to $r_1(\langle\phi, \kappa\rangle)$. To illustrate, let $[\![\alpha]\!] = \{\langle a, b\rangle, \langle b, b\rangle\}$ and $\sigma = \Box\bigcirc$. Then $[\![\sigma]\!]_D = D^2$ and $[\![\alpha]\!] \cap [\![\sigma]\!]_D = [\![\alpha]\!]$. We then form $D \otimes [\![\alpha]\!]$, a three place relation. Let $\kappa = \langle xyx\rangle$. Performing the respective cut by eliminating the positions corresponding to x yields $r_1(\langle\alpha, \langle xyx\rangle) = \{\langle a\rangle, \langle b\rangle\}$ and $r_1(\langle xyx\rangle) = \langle y\rangle$.[1]

The general definition now reads as follows:

(4) d. Let $\phi = \langle\alpha, \sigma\rangle \in Fml_n$, k a symbol, and $\kappa \in k@\sigma$. Then
  (i) $\langle\exists\kappa\phi, r_1(\kappa)\rangle \in Fml_m$ where m is the arity of $r_1(\kappa)$ and
  (ii) $[\![\langle\exists\kappa\phi, r_1(\kappa)\rangle]\!] = \langle r_1(\langle D \otimes ([\![\alpha]\!] \cap [\![\sigma]\!]_D), \kappa\rangle), [\![r_1(\kappa)]\!]_D\rangle$.

To illustrate, let us calculate the analogue of $\exists x R(x, x)$, assuming that $[\![R]\!] = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, a\rangle\}$. Intersection with $[\![xx]\!]_D$ (or $[\![\equiv]\!]$) yields $\{\langle a, a\rangle\}$ and $D \otimes \{\langle a, a\rangle\} = \{\langle a, a, a\rangle, \langle b, a, a\rangle, \ldots\}$. Since x is targeted, all positions must go, resulting in the zero-place relation $\{\emptyset\}$. Intersection with the empty sequence $\{\emptyset\}$ from (2) yields $\{\emptyset\}$ and the formula is true. Suppose $[\![R]\!] = \{\langle a, b\rangle, \langle a, c\rangle\}$. Intersection with identity yields $\emptyset$. $D \otimes \emptyset = \emptyset$ and $\emptyset \cap \{\emptyset\} = \emptyset$, hence the formula is false.

One of the reviewers suggests that the reader would benefit from "a concrete theorem on the equivalence of the new logic to classical first order logic". I will briefly review some of the formal results first published in Klein and Sternefeld (2017). All proofs can be found there; here, I will only sketch the main ideas.

First note that there is a straightforward way to translate from PL to C+I-PL and vice versa. Since alphabetic invariance introduces indeterminacy, we have to introduce some desambiguating conventions; without loss of generality both languages should agree on their use of variables. It is clear, then, that $P(x, y, z)$ unambiguously translates as $\langle P, \langle xyz\rangle\rangle$ (written as $T(P(x, y, z))$, and vice versa. Given the identity of variables, it is easy to define translations T and T^- by induction on the complexity of expressions of both languages.

Second, we observe a correspondence between the interpretation of variables. In PL and C+PL, assignment functions interpret variables, and by the coincidence theorem it is clear that we can reduce these functions to the parts that only interpret the free variables of a formula. These parts then directly conform to the sequences in $[\![\sigma]\!]_D$. Assuming that A is a formula of C-PL, we can define this correspondence as in (5-b):

(5) a. $g(\langle x_1, x_2, ..., x_n\rangle) := \langle g(x_1), g(x_2), ..., g(x_n)\rangle$.
  b. $GtoS(A) := \{s : \text{there is a } g \in [\![A]\!] \text{ such that } s = g(\pi_2(T(A)))\}$

(6) Lemma: $GtoS(A) = \pi_1([\![T(A)]\!]) \cap \pi_2([\![T(A)]\!])$

[1] This and all other operations in this section were recursively defined in Klein and Sternefeld (2017) by induction on the length of expressions. In this paper I prefer an informal presentation; the required definitions can be adopted from there.

The proof procedes by induction over the complexity of formulas.

Finally, it can be shown that A is satisfiable (true/false) iff $T(A)$ is satisfiable (true/false). Consider satisfiability:

(7) Proof: A is satisfiable iff $[\![A]\!] \neq \emptyset$ iff $GtoS(A) \neq \emptyset$ iff $\pi_1([\![T(A)]\!]) \cap \pi_2([\![T(A)]\!]) \neq \emptyset$ iff $T(A)$ is satisfiable.

See Klein and Sternefeld (2017) for further discussion.

4 Local symmetry

Having achieved alphabetic innocence and compositionality, the new implementation C+I-PL still shares an objectionable property with PL and C-PL. As a consequence of relying on sequences of variables and on denotations as sequences of individuals, the extensions of $(P(x) \wedge Q(y))$ and $(Q(y) \wedge P(x))$ differ, and so do the denotations $P \otimes Q$ and $Q \otimes P$ in C+I-PL. This has been discussed as the problem of *local asymmetry* in Klein and Sternefeld (2017). In this paper, I will discuss a new solution to the problem.[2]

Clearly, the problem of local asymmetry is related to the fact that the syntax of simple predicates already determines a certain ordering of arguments that is mapped into its semantics. As for complex predicates (generated by conjunction), this ordering is simply inherited by the ordering of the conjuncts. From a linguistic point of view, the interpretation of lexical items as implying a specific ordering is problematic in as far as the order of arguments of a predicate, viewed as the translation of an expression of natural language, could be language specific and arguably should not be encoded in its semantics. A pure semantics for natural language should therefore dispense with a pregiven order, the latter being a matter of syntax alone (cf. section 4.5. in Zimmermann and Sternefeld (2013)). But then, with the ordering of arguments being of no avail in pure semantics, $(P \wedge Q)$ and $(Q \wedge P)$ should also denote the same objects.

In linguistics, a standard order free semantics is provided by the theory of states and events: Each predicate P comes along with a state or event e and a set of n different thematic roles θ_i that specify a certain thematic relation (agent, patient, beneficient, etc.) between an individual and a state or event (cf. Parsons (1990)). Part of the theory (called *linking theory*) maps thematic relations onto a linear order that mirrors the order of arguments in natural language (and, by analogy, in standard predicate logic). The point is that the ontology of predicates does not implicate any ordering of their arguments.

As a consequence, the formulas in (8) all display the proposition that x beats y:

[2] A different, but minor problem might be seen in the compositionality of the coordination schemes. By using methods discussed in Quine (1960), sequences of symbols, although potentially infinite, can be defined recursively, as can their semantics. This is not directly expressed by the semantic rules stated above, but in principle, this could be added in order to make for a fully compositional semantics for all operations on σ and their denotations.

(8) a. $\exists e(\textsc{beat}(e) \wedge \textsc{agent}(x,e) \wedge \textsc{patient}(y,e))$
b. $\exists e(\textsc{beat}(e) \wedge \textsc{patient}(y,e) \wedge \textsc{agent}(x,e))$
c. $\exists e(\textsc{patient}(y,e) \wedge \textsc{agent}(x,e) \wedge \textsc{beat}(e))$
d. $\exists e(\textsc{agent}(x,e) \wedge \textsc{beat}(e) \wedge \textsc{patient}(y,e))$
...

According to this perspective, the semantic specification of verbs at the lexical level does not encompass information regarding linear order. Instead, it is the language-specific rules of grammar that imply that only (8-d) qualifies as the correct ”logical form” of an English sentence.

However, upon closer examination of (8) as formulas requiring semantic interpretation, it becomes apparent that the denotations of the conjunctions still remain asymmetric across all versions of Predicate Logic discussed thus far. Consequently, the theory of events, in its current form, is unable to resolve this issue.

Nontheless, thematic roles such as *agent*, *theme*, and *patient* can contribute to defining an order-free semantics. One approach is to associate each n-place predicate with n distinct thematic roles specific to that predicate. These roles have been termed ”individual thematic roles,” contrasting with the broader ”thematic role types” mentioned earlier.

For instance, it has been argued that an object of ”like” is not simply a theme or a patient (or any other run of the mill role types), but rather a ”likee,” representing an individual thematic role of ”like.” Similarly, the entity typically appearing as the subject of ”like” can be referred to as the ”liker.” To a certain extent, this line of reasoning reflects a well-known difficulty for theorists of thematic role types to properly identify thematic relations.

The following schematic comparison shows how an ordinary predicate extension relates to a thematic one (taken from Zimmermann and Sternefeld (2013) p. 80):

(9) a. Ordinary: $I(P^n) = \{\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_n\rangle, \ldots\}$
b. Thematic: $I(P^n) = \{\{\langle \theta_1, x_1\rangle, \ldots, \langle \theta_n, x_n\rangle\}, \{\langle \theta_1, y_1\rangle, \ldots, \langle \theta_n, y_n\rangle\}, \ldots\}$

The sequences in (9-a) are effectively replaced by sets in (9-b). Each predicate comes along with its own set of individual thematic roles (the “biter”, the “bitee”, etc.). The thematic roles as such do not contain any information about the ordering of elements in (9-a). Therefore, the index n on the thematic role in (9-b) is independent from and not to be confused with the index n on the meta-linguistic variable in (9-a). Each n-place predicate P can thus be identified with its set of (pairwise distinct) thematic roles $\{\theta_1^P, \ldots, \theta_n^P\}$ whose interpretation I in a model is as shown in (9-b).[3]

Before delving into a semantic system based on (9-b), a word of caution is

[3]Note that inverse predicates, *e.g.* $R =$ *beat* and $R^{-} =$ *being-beaten*, cannot be analysed by a reordering of arguments; rather they are different predicates with different thematic roles. Since a *beatee* is necessarily a *being-beaten*, these predicates must be related to each other by meaning postulates.

in order. Since Dowty (1989), an individual thematic role of a verb is defined as the set S of all properties which the verb entails for a given argument position. We might assume here, that the theta roles of (9-b) are names for such sets. However, the semantics of these roles does not play any role in our system. In fact, the exact nature of individual thematic roles will be of no avail. Rather we think of thematic roles as primitives that simply serve as place markers associated with a predicate. Clearly, thematic roles in the sense of Dowty are derived concepts that already presuppose an existing inferential logic, whereas thematic roles as used above are primitives in our system, needed to establish an inferential system in the first place. While the term "thematic role" may sound somewhat misleading if considered solely as a syntactic device, I will continue to use it instead of "place marker", thereby acknowledging the conventional usage found in textbooks.

One criticism leveled against individual thematic roles, contrasting them with role types, is that only the latter can provide the necessary information for establishing linking rules that determine grammatical relations like subject, direct object, indirect object etc. Role types are seen as more generalizable compared to the specific nature of individual thematic roles. However, according to Dowty (1991), even role types fall short in defining grammatical relations. Dowty suggests a system of proto-types grounded in logical relations between sentences instead. Accordingly, the present use of individual thematic roles as syntactic objects is fully compatible with Dowty's system of proto-types.

In the initial version of this paper, I proposed a method for capturing the effects of identical variables through additional statements of the form $\theta_i \equiv \theta_j$. The intention was not to say that theta roles are identical; rather I wanted to signify that across all elements of a (possibly complex, conjoined) predicate, if $\langle \theta_i, a\rangle$ and $\langle \theta_j, b\rangle$ occur as elements of the predicate, and if $\theta_i \equiv \theta_j$, then $a = b$. Regrettably, this approach proved too simplistic. A perceptive reviewer highlighted a problem one encounters when coordinating identical predicates. For instance, the equivalent of $P(x) \land P(y)$ in C-I-PL translates to a two-place relation derived from the Cartesian product of P with itself. However, within our current framework, we cannot rely on such products as they inherently assume an ordering. Consequently, it appears that the only viable means of distinguishing between $P(x)$ and $P(y)$ is by introducing additional variables.

Let us thus posit that with each occurrence of a predicate, a set of variables is associated, and each thematic role of the predicate is allocated a variable. For example, given the interpretation in (9-b) and the set $\sigma = \{\langle \Box, \theta_1\rangle, \langle \bigcirc, \theta_2\rangle, ...\langle \Box, \theta_n\rangle\}$ we can now pair each theta role in (9-b) with its respective variable as shown in (10):

(10) $$[\![P, \sigma]\!] = \{\{\langle \Box, \theta_1, x_1\rangle, \langle \bigcirc, \theta_2, x_2\rangle, ..., \langle \Box, \theta_n, x_n\rangle\}, \{\langle \Box, \theta_1, y_1\rangle, \langle \bigcirc, \theta_2, y_2\rangle, ..., \langle \Box, \theta_n, y_n\rangle\}, ...\}$$

As one might expect, this denotation is satisfiable only if it contains an element $\{\langle \Box, \theta_1, z_1\rangle, \langle \bigcirc, \theta_2, z_2\rangle, ...\langle \Box, \theta_n, z_n\rangle\}$ such that $z_1 = z_n$.

With this understanding we can now start the recursion on formulas and denotations by defining elementary formulas as illustrated above:

(11) a. If $P = \{\theta_1^P, ..., \theta_n^P\}$ is an n-place predicate with n distint theta roles, $v_1, ..., v_n$ are variables not necessarily distinct, and σ a set $\{\langle v_1, \theta_1^P\rangle, ..., \langle v_n, \theta_n^P\rangle\}$, then $\langle P, \sigma\rangle$ is a formula.
b. $[\![\langle P, \sigma\rangle]\!]$ is the smallest set S such that if $\langle \theta_i^P, a\rangle \in I(P)$ and $\langle v_j, \theta_i^P\rangle \in \sigma$, then $\langle v_j, \theta_i^P, a\rangle \in S$.

Conjunction is easy to define as a straightforward analogue to the Cartesian product, now in terms of sets with variables that may distinguish positions:

(12) a. If ϕ and ψ are formulas, $(\phi \wedge \psi)$ is a formula.
b. $[\![(\phi \wedge \psi)]\!] = \{x \cup y : x \in [\![\phi]\!] \text{ and } y \in [\![\psi]\!]\}$

Note in passing that the analogue of $[\![(P(x) \wedge P(y))]\!]$ can now easily be defined and is indeed symmetric. *E.g.*, assume that $I(P) = \{\langle \theta^P, a\rangle, \langle \theta^P, b\rangle\}$. Then $(\langle P, x\rangle \wedge \langle P, y\rangle)$ translates into $(\langle P, \langle x, \theta_1^P\rangle\rangle \wedge \langle P, \langle y, \theta_1^P\rangle\rangle)$ and

(13) $$[\![(\langle P, \langle x, \theta_1^P\rangle\rangle \wedge \langle P, \langle y, \theta_1^P\rangle\rangle)]\!] = \{\{\langle x, \theta^P, a\rangle, \langle y, \theta^P, a\rangle\}, \{\langle x, \theta^P, a\rangle, \langle y, \theta^P, b\rangle\}, \{\langle x, \theta^P, b\rangle, \langle y, \theta^P, a\rangle\}, \{\langle x, \theta^P, b\rangle, \langle y, \theta^P, b\rangle\}\}$$

Concerning negation, we have to define the complement of a relation R. Previously, we did this by subtracting R from a universal domain D^n. In the present framework, we have to define this domain as the set of all possible interpretations that agree with R with respect to the theta roles and variables of the relation, but possibly differ with repect to the indidividuals assigned to the theta roles.

(14) $D(\alpha) = \{\langle v_1, \theta_1, x_1\rangle, \ldots, \langle v_n, \theta_n, x_n\rangle\}$ such that $x_i \in D$ and for some $y_1, \ldots, y_n$, $\{\langle v_1, \theta_1, y_1\rangle, \ldots, \langle v_n, \theta_n, y_n\rangle\} \in \alpha$

(15) a. If ϕ is a formula, $\neg\phi$ is a formula.
b. $[\![\neg\phi]\!] = D([\![\phi]\!]) \setminus [\![\phi]\!]$

To establish existential quantification, it is advantageous to initially delineate the impact of a single variable co-occurring with two distinct thematic roles. Essentially, this entails narrowing down a given relation ϕ to a subset that adheres to the condition of identical thematic role values. Formally, we introduce the v-reduction $R(v, \phi)$ defined as follows:

(16) $R(v, \phi) = \{\alpha \in \phi : \langle v, \theta, e\rangle \in \alpha \text{ iff } \langle v, \theta', e\rangle \in \alpha\}$

We proceed to define existential quantification primarily through v-reduction. The quantification turns out false iff the set defined by reduction is empty. Otherwise, the reduction contains values for v that support the truth of the formula. Having reduced the number of possible values of a relation, we will, in a second step, diminish the arity of the relation by eliminating identical values of identical variables. Hence, arity reduction works in the same way as in C+I-PL.

(17) a. If ϕ is a formula and v a variable, then $\exists v\phi$ is a formula.
b. $[\![\exists v\phi]\!] = \{\alpha$: there is a $\alpha' \in R(v, \phi)$ such that α results from α' by eliminating all occurences of $\langle v, \theta, e\rangle \in \alpha'\}$

As previously stated, reduction to arity zero will yield a true sentence if and only if it results in $\emptyset$. Conversely, if v-reduction fails and produces an empty set, arity reduction cannot be applied. Consequently, the set defined in (17-b) is empty, which means that the sentence is false.

It is easily seen that the definitions in (3) carry over to the new system C+S-PL. As before, ϕ is satisfiable iff its denotation is not empty, ϕ is false iff it is empty, and ϕ is true iff $D(\phi) \subseteq [\![\phi]\!]$.

5 Symmetry and alphabetic innocence

Let us recap the developments sketched in this paper. Initially, we noted that classical PL lacks both compositionality and alphabetical innocence. Subsequently, we considered a compositional alternative C-PL, which lacks innocence because variables form part of the ontology. We then developed a new system C+I-PL which is both alphabetically innocent and compositional, but lacks the property of symmetry. We contended that asymmetry arises from an implicit assumption of PL, namely that relations rely on a fixed ordering of its arguments. Seeking to avoid this ordering assumption, we replaced tuples with sets including individual theta roles, leading to the development of C+S-PL. However, in the process of refining this system, we found it necessary to incorporate variables into the denotations. Consequently, this system cannot maintain innocence in terms of alphabetical neutrality.

One might still ask whether a system C+S+I-PL is feasable. Reconsidering our motivation to introduce variables, the challenge was to distinguish $P(x)$ from $P(y)$ in a formula like $(\exists x)(\exists y)(x \neq y \land (P(x) \land P(y)))$. Without an ordering, we are left with sets only, yet we clearly need two distinct objects in that set. And it is precisely the job of variables to make this distinction. In other words, if we aim to exclude variables from the ontology, there must be *something* that could replace them. However, I do not see what a natural choice for this something could be. Consequently, symmetry is necessarily at odds with alphabetical innocence.

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